BIPRODUCTS AND TWO-COCYCLE TWISTS OF HOPF ALGEBRAS

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ABSTRACT. Let H be a Hopf algebra with bijective antipode over a field k and suppose that $R\#H$ is a bi-product. Then R is a bialgebra in the Yetter–Drinfel'd category $^H_H\mathcal{Y}D$. We describe the bialgebras $(R\#H)^{op}$ and $(R\#H)^o$ explicitly as bi-products $R^{\underline{op}}\#H^{op}$ and $R^{\underline{o}}\#H^o$ respectively where $R^{\underline{op}}$ is a bialgebra in $^{H^{op}}_{H^{op}}\mathcal{Y}D$ and $R^{\underline{o}}$ is a bialgebra in $^{H^o}_{H^o}\mathcal{Y}D$. We use our results to describe two-cocycle twist bialgebra structures on the tensor product of bi-products.

INTRODUCTION

In [10] the irreducible representations of a certain class of pointed Hopf algebras over a field k is parameterized by pairs of characters, or by characters. These Hopf algebras are two-cocycle twists $H = (U\otimes A)^\sigma$ of the tensor product of pointed Hopf algebras or quotients of them. The twist structures are in one-one correspondence with bialgebra maps $U \longrightarrow A^{op\ o}$. In many cases the pointed Hopf algebras U and A are bi-products. We are thus led to consider the multiplicative opposite $(R\#H)^{op}$ and the dual $(R\#H)^o$ of a bi-product $R\#H$. Generally the multiplicative opposite and dual of a bi-product is a bi-product. One purpose of this paper is to characterize $(R\#H)^{op}$ and $(R\#H)^o$ as bi-products when H has bijective antipode. Recall that pointed Hopf algebras have bijective antipodes.

Let H be a Hopf algebra over k with bijective antipode and suppose that $R\#H$ is a bi-product. Then R is a bialgebra in the category of Yetter–Drinfel'd modules $^H_H\mathcal{Y}D$. We construct a bialgebra $R^{\underline{op}}$ in

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the Yetter–Drinfel'd category $^{H^{op}}_{H^{op}}\mathcal{Y}D$ such that $(R\#H)^{op} \simeq R^{\underline{op}}\#H^{op}$. Likewise we construct a bialgebra $R^{\underline{o}}$ in the Yetter–Drinfel'd category $^{H^o}_{H^o}\mathcal{Y}D$ such that $(R\#H)^o \simeq R^{\underline{o}}\#H^o$. These bialgebra constructions are based on more general procedures: the construction of an algebra (respectively a coalgebra) $A^{\underline{op}}$ in $^{H^{op}}_{H^{op}}\mathcal{Y}D$ from an algebra (respectively a coalgebra) A in $^H_H\mathcal{Y}D$ and the construction of an algebra (respectively a coalgebra) $A^{\underline{o}}$ in $^{H^o}_{H^o}\mathcal{Y}D$ from an algebra (respectively a coalgebra) A in $^H_H\mathcal{Y}D$.

Important to us is the case $U = \mathfrak{B}(W)\#k[\Gamma]$ and $A = \mathfrak{B}(V)\#k[\Lambda]$, where Γ and Λ are abelian groups and $(\mathfrak{B}(W), \mathfrak{B}(V))$ are Nichols algebras in the categories $^{k[\Gamma]}_{k[\Gamma]}\mathcal{Y}D, ^{k[\Lambda]}_{k[\Lambda]}\mathcal{Y}D$ respectively. These are of primary interest in [10]. For these Hopf algebras an extensive class of bialgebra maps $U \longrightarrow A^{op\,o}$ can be given in terms of two linear forms $\tau : k[\Lambda]\otimes k[\Gamma] \longrightarrow k$ and $\beta : V\otimes W \longrightarrow k$. The forms can easily be produced and thus, in particular, our results provide a way of generating a large number of two-cocycle twist bialgebras, that is bialgebra maps $\mathfrak{B}(W)\#k[\Lambda] \longrightarrow (\mathfrak{B}(V)\#k[\Gamma])^{op\,o}$ without checking the relations of $\mathfrak{B}(W)$ or $\mathfrak{B}(V)$ which are unknown in general.

For finite abelian groups $\Gamma, V \in {}^{k[\Gamma]}_{k[\Gamma]}\mathcal{Y}D$ and finite-dimensional Nichols algebras $\mathfrak{B}(V)$, the dual of $\mathfrak{B}(V)\#k[\Gamma]$ was already computed in [4, Theorem 2.2]. The two-cocycles in Corollary 9.1 were determined in [2] for finite-dimensional Nichols algebras with known relations by explicitly checking the relations. However, in [2] the more general case when U is not coradically graded, that is of the form $\mathfrak{B}(W)\#k[\Lambda]$, was considered.

This paper is organized as follows. In Section 1 we deal with the somewhat extensive prerequisites for the paper. First we discuss notations for algebras, coalgebras, and their representations, and then review algebraic objects in the Yetter–Drinfel'd category $^H_H\mathcal{Y}D$ of a Hopf algebra H with bijective antipode in detail for the reader's convenience. This discussion is important for Sections 2 and 3 where we describe algebra, coalgebra, and bialgebra constructions in the Yetter–Drinfel'd categories and $^{H^{op}}_{H^{op}}\mathcal{Y}D$ and $^{H^o}_{H^o}\mathcal{Y}D$ based on the their counterparts in $^H_H\mathcal{Y}D$. These constructions are basic ingredients in the realization of the multiplicative opposite and dual of a bi-product as a bi-product.

Certain bilinear forms on objects in Yetter–Drinfel'd categories which play a role in our construction of our two-cocycle twist Hopf algebras are introduced and studied in Section 4. In Section 5 we consider morphisms of bi-products. The multiplicative opposite of a bi-product is characterized as a bi-product in Section 6 and the dual of a bi-product is characterized as a bi-product in Section 7.

We apply the main results of Sections 6 and 7 to describe certain two-cocycle twists on the tensor product $(T\#K)\otimes(R\#H)$ of bi-products in Section 8. Here K and H are Hopf algebras with bijective antipodes over k. We focus on the basic case when $T = \mathfrak{B}(W)$ and $R = \mathfrak{B}(V)$ are Nichols algebras on finite-dimensional Yetter–Drinfel'd modules. In the last section we consider the our basic case when $K = k[\Lambda]$ and $H = k[\Gamma]$ are group algebras of abelian groups. Results here fit nicely into the discussion of [10].

We denote the antipode of a Hopf algebra H over k by S. Any one of [1, 6, 7, 11] will serve as a Hopf algebra reference for this paper. Throughout k is a field and all vector spaces are over k. For vector spaces U and V we will drop the subscript k from $\mathrm{End}_k(V)$, $\mathrm{Hom}_k(U, V)$, and $U\otimes_k V$. We denote the identity map of V by I_V. For a non-empty subset S of the dual space V^* we let $S^\perp$ denote the subspace of V consisting of the common zeros of the functionals in S. For $p \in U^*$ and $u \in U$ we denote the evaluation of p on u by $p(u)$ or $<p, u>$.

1. PRELIMINARIES

A good deal of prerequisite material is needed for this paper. We discuss general notation first then review specific topics in detail.

For a group G we let $\widehat{G}$ denote the group of characters of G with values in k. $H = k[G]$ denotes the group algebra of G over k which is Hopf algebra arising in most applications in this paper. For a Hopf algebra H over k we denote the group of grouplike elements of H by $G(H)$ as usual.

Let (A, m, η) be an algebra over k, which we shall usually denote by A. Generally we represent algebraic objects defined on a vector space by their underlying vector space. Observe that (A, m^{op}, η) is an algebra over k, where $m^{op} = m\circ\tau_{A,A}$. We denote A with this algebra structure by A^{op} and write $a^{op} = a$ for elements of A^{op}. Thus $a^{op}b^{op} = (ba)^{op}$ for all $a, b \in A$. This notation is very useful for computations in Yetter–Drinfel'd categories discussed below involving certain algebra constructions. We denote the category of left (respectively right) A-modules and module maps by $_A\mathcal{M}$ (respectively $\mathcal{M}_A$). If $\mathcal{C}$ is a category, by abuse of notation we will write $C \in \mathcal{C}$ to indicate that C is an object of $\mathcal{C}$.

Let M be a left A-module. Then M^* is a right A-module under the transpose action which is given by $(m^*\cdot a)(m) = m^*(a\cdot m)$ for all $m^* \in M^*$, $a \in A$, and $m \in M$. Likewise if M is a right A-module

then M^* is a left A-module where $(a \cdot m^*)(m) = m^*(m \cdot a)$ for all $a \in A$, $m^* \in M^*$, and $m \in M$.

Suppose B is an algebra also over k, let N be a left B-module, and suppose that $\varphi : A \longrightarrow B$ is an algebra map. Then a linear map $f : M \longrightarrow N$ is φ-linear if $f(a \cdot m) = \varphi(a) \cdot f(m)$ for all $a \in A$ and $m \in M$. There is a way of expressing the last equation in terms of A-module maps. Note that N is a left A-module by pullback along φ. Thus f is φ-linear if and only if f is a map of left A-modules.

Let (C, Δ, ϵ) be a coalgebra over k, which we usually denote by C. At times it is convenient to denote the coproduct Δ by Δ_C. Generally we use a variant on the Heyneman-Sweedler notation for the coproduct and write $\Delta(c) = c_{(1)} \otimes c_{(2)}$ to denote $\Delta(c) \in C \otimes C$ for $c \in C$. Note that $(C, \Delta^{cop}, \epsilon)$ is a coalgebra over k, where $\Delta^{cop} = \tau_{C,C} \circ \Delta$. We let C^{cop} denote the vector space C with this coalgebras structure and sometimes write $c^{cop} = c$ for elements of C^{cop}. With this notation $c^{cop}_{(1)} \otimes c^{cop}_{(2)} = c_{(2)} \otimes c_{(1)}$ for all $c \in C^{cop}$.

Suppose that (M, δ) is a left C-comodule. For $m \in M$ we use the notation $\delta(m) = m_{(-1)} \otimes m_{(0)}$ to denote $\delta(m) \in C \otimes M$. If (M, δ) is a right C-comodule we denote $\delta(m) \in M \otimes C$ by $\delta(m) = m_{(0)} \otimes m_{(1)}$. Observe that our coproduct and comodule notations do not conflict.

We make an exception to our coproduct notation described above for coalgebras in Yetter–Drinfel'd categories, in which case we write $\Delta(c) = c^{(1)} \otimes c^{(2)}$ for $c \in C$. See Section 1.2.

Suppose that M is a left C-comodule, D is a coalgebra over k, N is a left D-comodule, and $\varphi : C \longrightarrow D$ is a coalgebra map. Then a linear map $f : M \longrightarrow N$ is left φ-colinear if $\varphi(m_{(-1)}) \otimes f(m_{(0)}) = f(m)_{(-1)} \otimes f(m)_{(0)}$ for all $m \in M$. There is a way of expressing the last equation in terms of D-comodule maps. Note that M is a left D-comodule by pushout along φ. Thus f is φ-colinear if and only if f is a map of left D-modules. We use the terminology φ-linear and colinear as shorthand for φ-linear and φ-colinear.

Bilinear forms play an important role in this paper. We will think of them in terms of linear forms $\beta : U \otimes V \longrightarrow k$ and will often write $\beta(u, v)$ for $\beta(u \otimes v)$. Note that β determines linear maps $\beta_\ell : U \longrightarrow V^*$ and $\beta_r : V \longrightarrow U^*$ where $\beta_\ell(u)(v) = \beta(u, v) = \beta_r(v)(u)$ for all $u \in U$ and $v \in V$. The form β is left (respectively right) non-singular if β_ℓ (respectively β_r) is one-one and β is non-singular if it is both left and right non-singular.

For subspaces $X \subseteq U$ and $Y \subseteq V$ we define subspaces $X^\perp \subseteq V$ and $Y^\perp \subseteq U$ by

$$X^\perp = \{v \in V \mid \beta(X, v) = (0)\} \quad \text{and} \quad Y^\perp = \{u \in U \mid \beta(u, Y) = (0)\}.$$

Note that there is a form $\overline{\beta} : U/V^\perp \otimes V/U^\perp \longrightarrow k$ uniquely determined by $\overline{\beta} \circ (\pi_{V^\perp} \otimes \pi_{U^\perp}) = \beta$, where $\pi_{V^\perp} : U \longrightarrow U/V^\perp$ and $\pi_{U^\perp} : U \longrightarrow V/U^\perp$ are the projections. Observe that $V^\perp = \operatorname{Ker} \beta_\ell$, $U^\perp = \operatorname{Ker} \beta_r$, and that $\overline{\beta}$ is non-singular.

1.1. **Two-Cocycle Twist Bialgebras.** Let A be a bialgebra over k. A two-cocycle for A is a convolution invertible linear form $\sigma : A \otimes A \longrightarrow k$ which satisfies

$$\sigma(x_{(1)}, y_{(1)}) \sigma(x_{(2)} y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)}) \sigma(x, y_{(2)} z_{(2)})$$

for all $x, y, z \in A$. If σ is a two-cocycle for A then A^σ is a bialgebra, where $A^\sigma = A$ as a coalgebra and multiplication $m^\sigma : A \otimes A \longrightarrow A$ is given by

$$m^\sigma(x \otimes y) = \sigma(x_{1)}, y_{(1)}) x_{(2)} y_{(2)} \sigma^{-1}(x_{(3)}, y_{(3)})$$

for all $x, y \in A$.

Let U and A be bialgebras over k and suppose that $\tau : U \otimes A \longrightarrow k$ is a linear form. Consider the axioms:

(A.1) $\tau(u, aa') = \tau(u_{(2)}, a) \tau(u_{(1)}, a')$ for all $u \in U$ and $a, a' \in A$;
(A.2) $\tau(1, a) = \epsilon(a)$ for all $a \in A$;
(A.3) $\tau(uu', a) = \tau(u, a_{(1)}) \tau(u', a_{(2)})$ for all $u, u' \in U$ and $a \in A$;
(A.4) $\tau(u, 1) = \epsilon(u)$ for all $u \in U$.

Axioms (A.1)–(A.4) are equivalent to

(1.1) $\tau_\ell(U) \subseteq A^o$ and $\tau_\ell : U \longrightarrow A^{o\,cop} = A^{op\,o}$ is a bialgebra map

and they are also equivalent to

(1.2) $\tau_r(A) \subseteq U^o$ and $\tau_r : A \longrightarrow U^{o\,op}$ is a bialgebra map.

Suppose that (A.1)–(A.4) hold, τ is convolution invertible, and define a linear form $\sigma : (U \otimes A) \otimes (U \otimes A) \longrightarrow k$ by $\sigma(u \otimes a, u' \otimes a') = \epsilon(a) \tau(u', a) \epsilon(a')$ for all $u, u' \in U$ and $a, a' \in A$. Then σ is a two-cocycle. We denote the two-cocycle twist bialgebra structure on the tensor product bialgebra $U \otimes A$ by $H = (U \otimes A)^\sigma$. Observe that

$$(u \otimes a)(u' \otimes a') = u \tau(u'_{(1)}, a_{(1)}) u'_{(2)} \otimes a_{(2)} \tau^{-1}(u'_{(3)}, a_{(3)}) a'$$

for all $u, u' \in U$ and $a, a' \in A$.

Suppose that (A.1)–(A.4) hold for the linear form $\tau : U \otimes A \longrightarrow k$. Then τ is invertible if U has an antipode S or if A^{op} has an antipode ς. In the first case $\tau^{-1}(u, a) = \tau(S(u), a)$, and in the second $\tau^{-1}(u, a) = \tau(u, \varsigma(a))$, for all $u \in U$ and $a \in A$. See [10, Lemma 1.2].

As noted in [10, Section 1], the quantum double provides an important example of a two-cocycle twist bialgebra. This example is

described in [5] where two-cocycle twist bialgebras are defined and discussed.

Let U, $\overline{U}$ and A, $\overline{A}$ be algebras over k. Suppose further that $\tau : U \otimes A \longrightarrow k$ and $\overline{\tau} : \overline{U} \otimes \overline{A} \longrightarrow k$ are convolution invertible linear forms satisfying (A.1)–(A.4). Set $H = (U \otimes A)^\sigma$ and $\overline{H} = (\overline{U} \otimes \overline{A})^{\overline{\sigma}}$. Suppose that $f : U \longrightarrow \overline{U}$ and $g : A \longrightarrow \overline{A}$ are bialgebra maps such that $\overline{\tau}(f(u), g(a)) = \tau(u, a)$ for all $u \in U$ and $a \in A$. Then $f \otimes g : H \longrightarrow \overline{H}$ is a bialgebra map.

1.2. Yetter–Drinfel'd Categories and Their Algebras, Coalgebras, and Bialgebras.
Here we organize well-known material for the reader's convenience and for our use in later sections. See [3] in particular.

Let H be a bialgebra over k and let $^H_H \mathcal{Y}D$ be the category whose objects are triples $(M, \cdot, \delta)$, where $(M, \cdot)$ is a left H-module, (M, δ) is a left H-comodule, compatible in the sense that

$$(1.3) \qquad h_{(1)} m_{(-1)} \otimes h_{(2)} \cdot m_{(0)} = (h_{(1)} \cdot m)_{(-1)} h_{(2)} \otimes (h_{(1)} \cdot m)_{(0)}$$

for all $h \in H$ and $m \in M$, and whose morphisms $(M, \cdot, \delta) \longrightarrow (M', \cdot', \delta')$ are maps $f : M \longrightarrow M'$ simultaneously of left H-modules and of left H-comodules. We follow the convention of referring to an object of $^H_H \mathcal{Y}D$ as a Yetter–Drinfel'd module [3]. If H has an antipode S then (1.3) is equivalent to

$$(1.4) \qquad\qquad \delta(h \cdot m) = h_{(1)} m_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)}$$

for all $h \in H$ and $m \in M$, in practice a very useful formulation of the compatibility condition (1.3).

The category $^H_H \mathcal{Y}D$ has a monoidal structure, where k is given the left H-module structure $h \cdot 1_k = \epsilon(h)$ for all $h \in H$ and left H-comodule structure determined by $\delta(1_k) = 1_H \otimes 1_k$, and the tensor product of objects $M, N \in ^H_H \mathcal{Y}D$ is $M \otimes N$ as a vector space with left H-module structure given by $h \cdot (m \otimes n) = h_{(1)} \cdot m \otimes h_{(2)} \cdot n$ and left H-comodule structure given by $\delta(m \otimes n) = m_{(-1)} n_{(-1)} \otimes (m_{(0)} \otimes n_{(0)})$ for all $h \in H$, $m \in M$, and $n \in N$. When H is a Hopf algebra $^H_H \mathcal{Y}D$ is a braided monoidal category with braiding $\sigma_{M,N} : M \otimes N \longrightarrow N \otimes M$ for objects $M, N \in ^H_H \mathcal{Y}D$ determined by $\sigma_{M,N}(m \otimes n) = m_{(-1)} \cdot n \otimes m_{(0)}$ for all $m \in M$ and $n \in N$.

Let (A, m, η) be an algebra in $^H_H \mathcal{Y}D$. Then (A, m^{op}, η) is a well, where $m^{op} = m \circ \sigma_{A,A}$. Thus $a^{op} b^{op} = (a_{(-1)} \cdot b) a_{(0)}$ for all $a, b \in A$. We denote the object A with this algebra structure by A^{op}. If B is also an algebra in $^H_H \mathcal{Y}D$ then $A \otimes B$ is an algebra in $^H_H \mathcal{Y}D$, where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and $m_{A \otimes B} = (m_A \otimes m_B) \circ (I_A \otimes \sigma_{A,B} \otimes I_B)$. We write $A \underline{\otimes} B$ for $A \otimes B$ with

this algebra structure and $a\underline{\otimes}b = a\otimes b$ for tensors. By definition

$$(a\underline{\otimes}b)(a'\underline{\otimes}b') = a(b_{(-1)}\cdot a')\underline{\otimes}b_{(0)}b'$$

for all $a, a' \in A$ and $b, b' \in B$. Observe that the object k with its usual k-algebra structure is an algebra in $^H_H\mathcal{Y}D$.

Suppose (C, Δ, ϵ) is a coalgebra in $^H_H\mathcal{Y}D$. We shall write $\Delta(c) = c^{(1)}\otimes c^{(2)}$ for $c \in C$. Observe that $(C, \Delta^{cop}, \epsilon)$ is a coalgebra in $^H_H\mathcal{Y}D$, where $\Delta^{cop} = \sigma_{C,C}\circ\Delta$, or equivalently $\Delta^{cop}(c) = c^{(1)}_{(-1)}\cdot c^{(2)}\otimes c^{(1)}_{(0)}$ for all $c \in C$. We denote the object C with this coalgebra structure by C^{cop}. If D is a coalgebra in $^H_H\mathcal{Y}D$ also then $C\otimes D$ is a coalgebra in $^H_H\mathcal{Y}D$, where $\epsilon_{C\otimes D} = \epsilon_C\otimes\epsilon_D$ and $\Delta_{C\otimes D} = (I_C\otimes\sigma_{C,D}\otimes I_D)\circ(\Delta_C\otimes\Delta_D)$. We write $C\overline{\otimes}D$ for $C\otimes D$ with this coalgebra structure. By definition

$$\Delta(c\overline{\otimes}d) = (c^{(1)}\overline{\otimes}c^{(2)}_{(-1)}\cdot d^{(1)})\otimes(c^{(2)}_{(0)}\overline{\otimes}d^{(2)})$$

for all $c \in C$ and $d \in D$. Observe that the object k with its usual k-coalgebra structure is a coalgebra in $^H_H\mathcal{Y}D$.

Let $R \in {}^H_H\mathcal{Y}D$ be an algebra and a coalgebra in the category. Then $\Delta : R \longrightarrow R\underline{\otimes}R$ and $\epsilon : R \longrightarrow k$ are algebra maps if and only if $m : R\overline{\otimes}R \longrightarrow R$ and $\eta : k \longrightarrow R$ are coalgebra maps in which case we say that R with its algebra and coalgebra structure is a bialgebra in $^H_H\mathcal{Y}D$. If R is a bialgebra in $^H_H\mathcal{Y}D$ then R^{op}, R^{cop}, and therefore $R^{op\,cop}$, are bialgebras in $^H_H\mathcal{Y}D$. Observe that the object k with its usual k-algebra and k-coalgebra structure is a bialgebra in $^H_H\mathcal{Y}D$. If R and T are bialgebras in $^H_H\mathcal{Y}D$ then the object $R\otimes T$ with its algebra structure $R\underline{\otimes}T$ and coalgebra structure $R\overline{\otimes}T$ is a bialgebra in $^H_H\mathcal{Y}D$.

The most important bialgebra in $^H_H\mathcal{Y}D$ for us is the Nichols algebra. Let $M \in {}^H_H\mathcal{Y}D$ and consider the tensor k-algebra $T(M) = k\oplus M\oplus(M\otimes M)\oplus\cdots = \bigoplus_{\ell=0}^{\infty} M^{\otimes\ell}$ on the vector space M. Regard $T(M)$ as an object of $^H_H\mathcal{Y}D$, where

$$h\cdot(m_1\otimes\cdots\otimes m_r) = h_{(1)}\cdot m_1\otimes\cdots\otimes h_{(r)}\cdot m_r$$

for all $h \in H$ and $m_1, \ldots, m_r \in M$ and

$$\delta(m_1\otimes\cdots\otimes m_r) = m_{1\,(-1)}\cdots m_{r\,(-1)}\otimes(m_{1\,(0)}\otimes\cdots\otimes m_{r\,(0)})$$

for all $m_1, \ldots, m_r \in M$, describe the left H-module and left H-comodule structures respectively. Let $i : M \longrightarrow T(M)$ be the inclusion. Then the pair $(i, T(M))$ satisfies the obvious analog in $^H_H\mathcal{Y}D$ of the universal mapping property of the tensor algebra of a vector space as a k-algebra. Observe that $T(M) = \bigoplus_{\ell=0}^{\infty}M^{\otimes\ell}$ is a graded bialgebra (indeed Hopf algebra) in $^H_H\mathcal{Y}D$.

The algebra $T(M)$ of $^H_H\mathcal{Y}D$ is a bialgebra in the category just as the tensor algebra of a vector space is a k-bialgebra. The linear maps $d :$

$M \longrightarrow T(M)\underline{\otimes}T(M)$ and $e : M \longrightarrow k$ defined by $d(m) = 1\underline{\otimes}m + m\underline{\otimes}1$ and $e(m) = 0$ respectively for all $m \in M$ lift to algebra morphisms $\Delta : T(M) \longrightarrow T(M)\underline{\otimes}T(M)$ and $\epsilon : T(M) \longrightarrow k$ uniquely determined by $\Delta\circ i = d$ and $\epsilon\circ i = e$. The structure $(T(M), \Delta, \epsilon)$ is a coalgebra in ${}^H_H\mathcal{Y}D$ and $T(M)$ with its algebra and coalgebra structure is a bialgebra in ${}^H_H\mathcal{Y}D$. The pair $(i, T(M))$ satisfies the following universal mapping property: If A is a bialgebra in ${}^H_H\mathcal{Y}D$ and $f : M \longrightarrow A$ is a morphism such that Im $f \subseteq P(A)$ then there is a bialgebra morphism $F : T(M) \longrightarrow A$ uniquely determined by $F\circ i = f$.

The Nichols algebra $\mathfrak{B}(M)$ has a very simple theoretical description. Among the graded subobjects J of $\mathfrak{B}(M)$ which are coideals and satisfy $J\cap M = (0)$, there is a unique maximal one I. It is easy to see that I is an ideal of $\mathfrak{B}(M)$; thus the subobject I is a bi-ideal. Consequently the quotient $\mathfrak{B}(M) = T(M)/I$ is a connected graded bialgebra in the category ${}^H_H\mathcal{Y}D$. Observe that $\mathfrak{B}(M)$ is generated as an algebra by $\mathfrak{B}(M)(1) = M$, which is also the space of primitive elements of $\mathfrak{B}(M)$. Since $I\cap\mathfrak{B}(M) = (0)$ we may think of M as a subspace of $\mathfrak{B}(M)$. The pair $(M, (\mathfrak{B}(M))$ satisfies the following universal mapping property:

Theorem 1.1. *Let H be a Hopf algebra and $M \in {}^H_H\mathcal{Y}D$. Then:*

  a) $\mathfrak{B}(M)$ *is a connected graded bialgebra in ${}^H_H\mathcal{Y}D$ and $M = \mathfrak{B}(1)$ is a subobject which generates $\mathfrak{B}(M)$ as an algebra.*
  b) *If A is a connected graded bialgebra in ${}^H_H\mathcal{Y}D$ generated as an algebra by $A(1)$ and $f : A(1) \longrightarrow M$ is a morphism, then there is a bialgebra morphism $F : A \longrightarrow \mathfrak{B}(M)$ determined by $F|_{A(1)} = f$.*

PROOF: We need only show part b). By the universal mapping property of the bialgebra $(T(A(1)), i)$ there is a bialgebra morphism $F : T(A(1)) \longrightarrow A$ determined by $F|_{A(1)} = i$. Since $T(A(1))$ is generated by $A(1)$ as an algebra, F is an onto morphism of graded bialgebras. Let $J = \text{Ker } F$. Then J is a sub-object of $T(A(1))$ which is a graded bi-ideal of $T(A(1))$ satisfying $J\cap A(1) = (0)$. Using the universal mapping property again we see that the morphism $f : A(1) \longrightarrow M$ induces a bialgebra morphism $T(f) : T(A(1)) \longrightarrow T(M)$ determined by $T(f)|_{A(1)} = f|_{A(1)}$. Observe that $T(f)(J)$ is a subobject of $T(M)$ which is a graded bi-ideal of $T(M)$ whose intersection with M is (0). This means $T(f)(J) \subseteq I$, where the latter is defined above. The composite $A \simeq T(A(1))/J \longrightarrow T(M)/I = \mathfrak{B}(M)$, where the second map is defined by $x + J \mapsto T(f)(x) + I$, is our desired bialgebra morphism F. $\square$

We have noted that $\mathfrak{B}(M)(1)$ is the subspace of primitive elements of $\mathfrak{B}(M)$. Thus $\mathfrak{B}(M)$ is a connected graded primitively generated bialgebra in $^H_H\mathcal{Y}D$ with subspace of primitive elements $\mathfrak{B}(M)(1)$. These are defining properties.

Corollary 1.2. *Let H be a Hopf algebra over the field k and suppose that A is a connected graded primitively generated bialgebra in $^H_H\mathcal{Y}D$ with subspace of primitive elements $A(1)$. Then there is an isomorphism of bialgebras $A \simeq \mathfrak{B}(A(1))$ which extends the identity map $I_{A(1)}$.*

PROOF: Let $F : A \longrightarrow \mathfrak{B}(A(1))$ be the bialgebra morphism of part b) of Theorem 1.1 which extends $I_{A(1)}$. Since $A(1)$ generates $\mathfrak{B}(A(1))$ the map F is onto. Now $\mathrm{Ker}\, F \cap P(A) = \mathrm{Ker}\, F \cap A(1) = (0)$. Generally if C is a connected coalgebra and $f : C \longrightarrow C'$ is a coalgebra map which satisfies $\mathrm{Ker}\, f \cap P(C) = (0)$ then f is one-one [11, Lemma 11.0.1]. Thus the onto map F is one-one. $\square$

We leave the reader with the exercise of verifying the following corollary to the theorem above.

Corollary 1.3. *Let K and H be Hopf algebras with bijective antipodes over the field k and let $\varphi : K \longrightarrow H$ be a bialgebra map. Suppose that $W \in {}^K_K\mathcal{Y}D$, $W \in {}^H_H\mathcal{Y}D$, and $f : W \longrightarrow V$ is φ-linear and colinear. Then:*

   a) *There is a map of algebras and coalgebras $\mathfrak{B}(f) : \mathfrak{B}(W) \longrightarrow \mathfrak{B}(V)$ determined by $\mathfrak{B}(f)|_W = f$. Furthermore $\mathfrak{B}(f)$ is φ-linear and colinear.*
   b) *If f is one-one (respectively onto) then $\mathfrak{B}(f)$ is one-one (respectively onto).*

$\square$

2. Associated Constructions in $^{H^{op}}_{H^{op}}\mathcal{Y}D$

Throughout this section H has bijective antipode S. Starting with objects, algebras, coalgebras, and bialgebras in $^H_H\mathcal{Y}D$ we construct counterparts in $^{H^{op}}_{H^{op}}\mathcal{Y}D$ which are important for the analysis of biproducts in Section 8. First we start with objects.

Let $(M, \cdot, \delta) \in {}^H_H\mathcal{Y}D$. Then $(M, \cdot_{op}, \delta) \in {}^{H^{op}}_{H^{op}}\mathcal{Y}D$, where

$$(2.1) \qquad\qquad h \cdot_{op} m = S^{-1}(h) \cdot m$$

for all $h \in H$ and $m \in M$. We denote $(M, \cdot_{op}, \delta)$ by $M^{\underline{op}}$. If N is also an object of $^H_H\mathcal{Y}D$ and $f : M \longrightarrow N$ is a morphism, then $f^{\underline{op}} : M^{\underline{op}} \longrightarrow N^{\underline{op}}$ is a morphism, where $f^{\underline{op}} = f$.

When M has the structure of an algebra, coalgebra, or bialgebra, then $M^{\underline{op}} \in {}^{H^{op}}_{H^{op}}\mathcal{YD}$ does as well. If (A, m, η) is an algebra in ${}^{H}_{H}\mathcal{YD}$ then $(A^{\underline{op}}, m^{\underline{op}}, \eta)$ is an algebra in ${}^{H^{op}}_{H^{op}}\mathcal{YD}$, where

$$(2.2) \qquad\qquad m^{\underline{op}}(a \otimes b) = ba$$

for all $a, b \in A$. If $\overline{A}$ is also an algebra in ${}^{H}_{H}\mathcal{YD}$ and $f : A \longrightarrow \overline{A}$ is an algebra morphism then $f : A^{\underline{op}} \longrightarrow \overline{A}^{\underline{op}}$ is an algebra morphism. If (C, Δ, ϵ) is a coalgebra in ${}^{H}_{H}\mathcal{YD}$ then $(C^{\underline{op}}, \Delta^{\underline{op}}, \epsilon)$ is a coalgebra in ${}^{H^{op}}_{H^{op}}\mathcal{YD}$, where

$$(2.3) \qquad\qquad \Delta^{\underline{op}}(c) = c^{(2)}{}_{(-1)} \cdot^{op} c^{(1)} \otimes c^{(2)}{}_{(0)}$$

for all $c \in C$. If C' is also a coalgebra in ${}^{H}_{H}\mathcal{YD}$ and $f : C \longrightarrow \overline{C}$ is a coalgebra morphism then $f : C^{\underline{op}} \longrightarrow \overline{C}^{\underline{op}}$ is a coalgebra morphism. If $(R, m, \eta, \Delta, \epsilon)$ is a bialgebra in ${}^{H}_{H}\mathcal{YD}$ then $(R^{\underline{op}}, m^{\underline{op}}, \eta, \Delta^{\underline{op}}, \epsilon)$ is a bialgebra in ${}^{H^{op}}_{H^{op}}\mathcal{YD}$. If $\overline{R}$ is also a bialgebra in ${}^{H}_{H}\mathcal{YD}$ and $f : R \longrightarrow \overline{R}$ is a bialgebra morphism, then $f : R^{\underline{op}} \longrightarrow \overline{R}^{\underline{op}}$ is a bialgebra morphism.

Our assertions about $A^{\underline{op}}$, $C^{\underline{op}}$, and $R^{\underline{op}}$ can be shown directly with a good deal of effort. For this the a^{op} and c^{cop} notations are strongly recommended. A more illuminating approach which yields much easier proofs is to recognize that there is an isomorphism (F, ϑ) of the monoidal categories ${}^{H}_{H}\mathcal{YD}$ and ${}^{H^{op}}_{H^{op}}\mathcal{YD}$. The functor $F : {}^{H}_{H}\mathcal{YD} \longrightarrow {}^{H^{op}}_{H^{op}}\mathcal{YD}$ is defined by $F(M) = M^{\underline{op}}$ for objects M and $F(f) = f$ for morphisms f. The morphism of left H^{op}-modules $\vartheta_{M,N} : F(M \otimes N) \longrightarrow F(M) \otimes F(N)$ is defined by $\vartheta_{M,N}(m \otimes n) = S^{-1}(n_{(-1)}) \cdot m \otimes n_{(0)}$ for all $m \in M$ and $n \in N$. Observe that $\vartheta^{-1}_{M,N} : F(M) \otimes F(N) \longrightarrow F(M \otimes N)$ is given by $\vartheta^{-1}_{M,N}(m \otimes n) = n_{(-1)} \cdot m \otimes n_{(0)}$ for all $m \in M$ and $n \in N$.

Let (A, m, η) be an algebra in ${}^{H}_{H}\mathcal{YD}$. Then $(F(A), F(m) \circ \vartheta^{-1}_{A,A}, F(\eta))$ is an algebra of ${}^{H^{op}}_{H^{op}}\mathcal{YD}$. Since $m^{\underline{op}} = (F(m) \circ \vartheta^{-1}_{A,A})^{op}$ it follows that $A^{\underline{op}}$ is an algebra in ${}^{H^{op}}_{H^{op}}\mathcal{YD}$ as well. Let (C, Δ, ϵ) be a coalgebra of ${}^{H}_{H}\mathcal{YD}$. Then $(F(C), \vartheta_{C,C} \circ F(\Delta), F(\epsilon))$ is a coalgebra of ${}^{H^{op}}_{H^{op}}\mathcal{YD}$ which is $C^{\underline{op}}$. If R is a bialgebra in ${}^{H}_{H}\mathcal{YD}$ then the object $R^{\underline{op}}$ with its algebra and coalgebra structures $R^{\underline{op}}$ is a bialgebra in ${}^{H^{op}}_{H^{op}}\mathcal{YD}$.

Suppose that H has bijective antipode and let V by an object of ${}^{H}_{H}\mathcal{YD}$. Using Corollary 1.2 we are able to relate $\mathfrak{B}(V)^{\underline{op}}$ to a Nichols algebra.

Observe that the grading of $\mathfrak{B}(V)$ is a bialgebra grading for $\mathfrak{B}(V)^{\underline{op}}$. It is not hard to see that $\mathfrak{B}(V)(1)$ generates $\mathfrak{B}(V)^{\underline{op}}$ and is also the space of primitives of $\mathfrak{B}(V)^{\underline{op}}$. As a subobject of $\mathfrak{B}(V)^{\underline{op}}$ note that $\mathfrak{B}(V)^{\underline{op}}(1) = V^{\underline{op}}$. Thus

$$(2.4) \qquad\qquad \mathfrak{B}(V^{\underline{op}}) = \mathfrak{B}(V)^{\underline{op}}$$

by Corollary 1.2.

3. Associated Constructions in $^{H^o}_{H^o}\mathcal{Y}D$

We now turn to constructions in $^{H^o}_{H^o}\mathcal{Y}D$. As in the section H has bijective antipode S. Starting with objects, algebras, coalgebras, and bialgebras in $^H_H\mathcal{Y}D$ we construct counterparts in $^{H^o}_{H^o}\mathcal{Y}D$ which are important for the analysis of bi-products in Section 8.

First we consider the objects. Let $(M, \eta, \delta) \in {}^H_H\mathcal{Y}D$. We construct an object (M^r, δ^o, η^o) of $^{H^o}_{H^o}\mathcal{Y}D$. Regard $H^* \otimes M^*$ as a subspace of $(H \otimes M)^*$ in the usual way. Recall that M^r, the subspace of all $m^* \in M^*$ which vanish on $I \cdot M$ for some cofinite ideal I of H, can be characterized as $M^r = (\eta^*)^{-1}(H^* \otimes M^*)$. Furthermore $\eta^*(M^r) \subseteq H^o \otimes M^r$ and (M^r, η^o) is a left H^o-comodule, where $\eta^o = \eta^*|_{M^r}$. Thus the comodule action $\eta^o(m^*) = m^*_{(-1)} \otimes m^*_{(0)}$ of η^o on $m^* \in M^r$ is determined by

$$(3.1) \qquad m^*_{(-1)}(h)m^*_{(0)}(m) = m^*(h \cdot m)$$

for all $h \in H$ and $m \in M$. See [8] for example.

The left H-comodule structure (M, δ) induces a (rational) right H^*-module action in M which in turn induces a left H^*-module structure $(M^*, \cdot)$ on M^* under the transpose action. It is easy to see that $(M^*, \cdot) = (M^*, \delta^*|_{H^* \otimes M^*})$. By restriction of the H^*-action M^* is a left H^o-module. A straightforward calculation yields

$$\eta^*(h^o \cdot m^*) = h^o{}_{(1)} m^*_{(-1)} S(h^o{}_{(3)}) \otimes h^o{}_{(2)} \cdot m^*_{(0)}$$

for all $h^o \in H^o$ and $m^* \in M^r$, where S is the antipode of H^o. Let $\delta^o = \delta^*|_{H^o \otimes M^r}$. Thus $H^o \cdot M^r \subseteq M^r$; hence (M^r, δ^o) is an H^o-submodule of M^* which we denote $(M^r, \cdot)$. The equivalence of (1.3) and (1.4) imply that $(M^r, \cdot, \eta^o) \in {}^{H^o}_{H^o}\mathcal{Y}D$. The left H^o-module action on M^r is given explicitly by

$$(3.2) \qquad (h^o \cdot m^*)(m) = m^*(m \leftharpoonup h^o) = h^o(m_{(-1)})m^*(m_{(0)})$$

for all $h^o \in H^o$, $m^* \in M^r$, and $m \in M$. Note that $(M^{\underline{op}})^r = M^r$ as vector spaces. If N is also an object of $^H_H\mathcal{Y}D$ and $f : M \longrightarrow N$ is a morphism then $f^*(N^r) \subseteq M^r$ and the restriction $f^r = f^*|_{N^r}$ is a morphism $f^r : N^r \longrightarrow M^r$ since $f^*(n^*)(h \cdot n) = n^*_{(-1)}(h)f^*(n^*_{(0)})(n)$ for all $n^* \in N^r$, $h \in H$, and $n \in N$.

Suppose that (C, Δ, ϵ) is a coalgebra in $^H_H\mathcal{Y}D$. Then the object $C^r \in {}^{H^o}_{H^o}\mathcal{Y}D$ has the structure of an algebra in the category; as a k-algebra it is a subalgebra of the dual algebra C^*. We show that C^r is a subalgebra of C^* and leave the remaining details of the proof that C^r is an algebra in $^{H^o}_{H^o}\mathcal{Y}D$ to the reader.

Since $\epsilon : C \longrightarrow k$ is a morphism $\operatorname{Ker} \epsilon$ is a left H-submodule of C. Therefore $\epsilon \in C^r$. Suppose that $a, b \in C^r$. Then $a(I \cdot C) = (0) = b(J \cdot C)$ for some cofinite ideals I and J of H. Since Δ_H is an algebra map, $L = \Delta_H^{-1}(I \otimes H + H \otimes J)$ is an ideal of H which is cofinite since I and J are cofinite ideals of H. Since $\Delta = \Delta_C$ is a map of left H-modules it follows that $\Delta(h \cdot c) = h_{(1)} \cdot c^{(1)} \otimes h_{(2)} \cdot c^{(2)}$ for all $h \in H$ and $c \in C$. Thus

$$(ab)(K \cdot C) \subseteq (a \otimes b)(\Delta(K \cdot C)) \subseteq (a \otimes b)(I \cdot C \otimes C \cdot C + C \cdot C \otimes J \cdot C) = (0)$$

from which $ab \in C^r$ follows. Also note that if $\overline{C}$ is another coalgebra in ${}^H_H \mathcal{YD}$ and if $f : C \longrightarrow \overline{C}$ is a coalgebra morphism then $f^r : \overline{C}^r \longrightarrow C^r$ is an algebra morphism.

One can think of C^r as the counterpart in ${}^{H^o}_{H^o} \mathcal{YD}$ of the dual k-algebra C^*. Suppose A is an algebra in ${}^H_H \mathcal{YD}$. There is a counterpart $A^{\underline{o}}$ in ${}^{H^o}_{H^o} \mathcal{YD}$ to the dual k-coalgebra A^o which arises very naturally in Section 7. As a vector space $A^{\underline{o}}$ is the set of all functionals in A^* which vanish on a cofinite subspace I of A which is both an ideal of A and also a left H-submodule of A.

Now $I^\perp$ is a subcoalgebra of the dual k-coalgebra A^o and is also a left H-submodule of A^r. Since the intersection of two cofinite subspaces of A which are both ideals of A and left H-submodules of A has the same properties, it follows that $A^{\underline{o}}$ is a subcoalgebra of A^o and also a subobject of A^r. At this point is not hard to see that $A^{\underline{o}}$ is a coalgebra in ${}^{H^o}_{H^o} \mathcal{YD}$. Let $\overline{A}$ be an algebra in ${}^H_H \mathcal{YD}$ also and suppose that $f : A \longrightarrow \overline{A}$ is an algebra morphism. Then $f^r(\overline{A}^{\underline{o}}) \subseteq A^{\underline{o}}$ and the restriction $f^{\underline{o}} = f^r|_{\overline{A}^{\underline{o}}}$ is a coalgebra morphism $f^{\underline{o}} : \overline{A}^{\underline{o}} \longrightarrow A^{\underline{o}}$.

Suppose that R is a bialgebra in ${}^H_H \mathcal{YD}$. Then the object $R^{\underline{o}}$ of ${}^{H^o}_{H^o} \mathcal{YD}$ is a bialgebra in ${}^{H^o}_{H^o} \mathcal{YD}$ with the subalgebra structure of the k-algebra R^* and the subcoalgebra structure of the k-coalgebra R^o. Furthermore, if $\overline{R}$ is also a bialgebra in ${}^H_H \mathcal{YD}$ and $f : R \longrightarrow \overline{R}$ is a bialgebra morphism, then $f^{\underline{o}} : \overline{R}^{\underline{o}} \longrightarrow R^{\underline{o}}$ is a bialgebra morphism.

Let V be an object of ${}^H_H \mathcal{YD}$. There is a natural relationship between $\mathfrak{B}(V)^{\underline{o}}$ and a Nichols algebra. Consider the one-one map $i : V^r \longrightarrow \mathfrak{B}(V)^*$ defined for $v^* \in V^{\underline{o}}$ by

$$i(v^*) = \begin{cases} v^*(x) & : \quad x \in \mathfrak{B}(V)(1) = V \\ 0 & : \quad x \in \mathfrak{B}(V)(n), \ \ n \neq 1 \end{cases}.$$

Then $\operatorname{Im} i \subseteq \mathfrak{B}(V)^{\underline{o}}$ and $i : V^r \longrightarrow \mathfrak{B}(V)^{\underline{o}}$ is a one-one morphism. Let $\mathcal{I} : \mathfrak{B}(V^r) \longrightarrow \mathfrak{B}(V)^{\underline{o}}$ be the bialgebra morphism of Corollary 1.2 which extends i. Since $\operatorname{Ker} \mathcal{I} \cap P(\mathfrak{B}(V^r)) = \operatorname{Ker} \mathcal{I} \cap V^r = (0)$ it follows that $\mathcal{I}$ is one-one by [11, Lemma 11.0.1]. again. We have shown

$$(3.3) \qquad \mathcal{I} : \mathfrak{B}(V^r) \longrightarrow \mathfrak{B}(V)^{\underline{o}} \quad \text{is a one-one bialgebra morphism}$$

When V is finite-dimensional $\mathfrak{B}(V^r)$ is identified with the graded dual of $\mathfrak{B}(V)$ via the map $\mathcal{I}$. In the special case of a Yetter–Drinfel'd module over the group algebra of a finite group with finite-dimensional $\mathfrak{B}(V)$, the dual of $\mathfrak{B}(V)$ was determined in [4, Theorem 2.2].

4. Bilinear Forms in the Yetter–Drinfel'd Context

Let H be a Hopf algebra with bijective antipode. Let R, T be bialgebras in $^H_H\mathcal{Y}D$ and suppose that $\beta : T \otimes R \longrightarrow k$ is a linear form. We will find the following analogs to (A.1)–(A.4) useful:

(B.1) $\beta(tt', r) = \beta(t, S^{-1}(r^{(2)}{}_{(-1)}) \cdot r^{(1)}) \beta(t', r^{(2)}{}_{(0)})$ for all $t, t' \in T$ and $r \in R$;

(B.2) $\beta(1, r) = \epsilon(r)$ for all $r \in R$;

(B.3) $\beta(t, rr') = \beta(t^{(2)}, r) \beta(t^{(1)}, r')$ for all $t \in T$ and $r, r' \in R$;

(B.4) $\beta(t, 1) = \epsilon(t)$ for all $t \in T$.

We leave the reader with the exercise of establishing the equivalence of (B.1)–(B.4) with analogs of (1.1) and (1.2) :

Lemma 4.1. *Let H be a Hopf algebra with bijective antipode over the field k, let T and R be bialgebras in $^H_H\mathcal{Y}D$, and suppose $\beta : T \otimes R \longrightarrow k$ is a linear form. Then the following are equivalent:*

  a) *(B.1)–(B.4) hold.*
  b) *$\beta_\ell(T) \subseteq R^{\underline{op}\,\underline{o}}$ and $\beta_\ell : T \longrightarrow R^{\underline{op}\,\underline{o}}$ is a bialgebra map.*
  c) *$\beta_r(R) \subseteq T^{\underline{o}\,\underline{op}}$ and $\beta_r : R \longrightarrow T^{\underline{o}\,\underline{op}}$ is a bialgebra map.*

$\square$

Let K and H be bialgebras over k and suppose H has bijective antipode S. Let $W \in {}^K_K\mathcal{Y}D$, $V \in {}^H_H\mathcal{Y}D$, and let $\tau : K \otimes H \longrightarrow k$, $\beta : W \otimes V \longrightarrow k$ be linear forms. Two conditions relating τ and β will play an important part in this paper:

(C.1) $\beta(k \cdot w, v) = \beta(w, v \leftharpoonup \tau_\ell(k))$ for all $k \in K$, $w \in W$, and $v \in V$;

(C.2) $\beta(w \leftharpoonup \tau_r(h), v) = \beta(w, S^{-1}(h) \cdot v)$ for all $w \in W$, $h \in H$, and $v \in V$.

The first (C.1) implies $V^\perp = \operatorname{Ker} \beta_\ell$ is a K-submodule of W and the second (C.2) implies $W^\perp = \operatorname{Ker} \beta_r$ is an H-submodule of V. These conditions have formulations in terms of linear and colinear maps.

Proposition 4.2. *Let K and H be Hopf algebras with bijective antipodes over the field k. Suppose that $\tau : K \otimes H \longrightarrow k$ satisfies (A.1)–(A.4), let $W \in {}^K_K\mathcal{Y}D$, $V \in {}^H_H\mathcal{Y}D$, and let $\beta : W \otimes V \longrightarrow k$ is a linear form. Then the following are equivalent:*

  a) *(C.1) and (C.2) hold.*

  b) $\beta_\ell(W) \subseteq (V^{\underline{op}})^r$ and $\beta_\ell : W \longrightarrow (V^{\underline{op}})^r$ is τ_ℓ-linear and colinear.
  c) $\beta_r(V) \subseteq W^r$ and $\beta_r : V \longrightarrow W^r$ is τ_r-linear and colinear.

PROOF: We show that parts a) and b) are equivalent and leave the reader with the exercise of adapting our proof to establish the equivalence of parts a) and c). In the latter the roles of (C.1) and (C.2) are reversed.

Suppose that $\beta_\ell(W) \subseteq (V^{\underline{op}})^r$ and consider the linear map $\beta_\ell : W \longrightarrow (V^{\underline{op}})^r$. Using (3.2) it follows that β_ℓ is τ_ℓ-linear if and only if (C.1) holds. Using (3.1), where $S^{-1}(h) \cdot v = h \cdot_{op} v$ replaces $h \cdot v$, it follows that β_ℓ is τ_ℓ-colinear if and only if (C.2) holds.

Suppose that (C.2) holds. Now W is a right K^*-module under the rational action arising from (M, δ). Now $\tau_r : H \longrightarrow (K^o)^{op}$ is an algebra map by (1.2). Thus W is left H-module by pullback along τ_ℓ. Let $w \in W$. Then $H \cdot w = w \leftharpoonup \tau_r(H)$ is finite-dimensional, there is a cofinite ideal I of H such that $(0) = I \cdot w = w \leftharpoonup \tau_r(I)$. Thus

$$\beta_\ell(w)(I \cdot_{op} V) = \beta(w, S^{-1}(I) \cdot V) = \beta(w \leftharpoonup \tau_r(I), V) = (0)$$

which means that $\beta_\ell(w) \in (V^{\underline{op}})^r$. $\square$

Corollary 4.3. *Suppose that K and H are Hopf algebras with bijective antipodes over the field k and suppose that $\tau : K \otimes H \longrightarrow k$ satisfies (A.1)–(A.4). Let $W \in {}^K_K \mathcal{YD}$, $W \in {}^H_H \mathcal{YD}$, and τ and $\beta : W \otimes V \longrightarrow k$ satisfy (C.1) and (C.2). Then:*

  a) *There is a form $\mathfrak{B}(\beta) : \mathfrak{B}(W) \otimes \mathfrak{B}(V) \longrightarrow k$ determined by the properties that it satisfies (B.1)–(B.4) and $\mathfrak{B}(\beta)|_{W \otimes V} = \beta$. Furthermore $\mathfrak{B}(\beta)$ satisfies (C.1) and (C.2).*
  b) *Suppose that $\overline{K}$ and $\overline{H}$ are also Hopf algebras with bijective antipodes over k, $\overline{\tau} : \overline{K} \otimes \overline{H} \longrightarrow k$ satisfies (A.1)–(A.4), $W \in {}^{\overline{K}}_{\overline{K}} \mathcal{YD}$, $\overline{W} \in {}^{\overline{H}}_{\overline{H}} \mathcal{YD}$, and $\overline{\tau}$ and $\overline{\beta} : \overline{W} \otimes \overline{V} \longrightarrow k$ satisfy (C.1) and (C.2). If $\overline{\beta} \circ (f \otimes g) = \beta$ then $\mathfrak{B}(\overline{\beta}) \circ (\mathfrak{B}(f) \otimes \mathfrak{B}(g)) = \mathfrak{B}(\beta)$.*

$\square$

PROOF: By Proposition 4.2 b), $\beta_\ell : W \longrightarrow (V^{\underline{op}})^r$ is τ_ℓ-linear and colinear, and by Corollary 1.3

$$\mathfrak{B}(\beta_\ell) : \mathfrak{B}(W) \longrightarrow \mathfrak{B}((V^{\underline{op}})^r)$$

is a bialgebra map. Then we define $\mathfrak{B}(\beta)_\ell$ as the composition of $\mathfrak{B}(\beta_\ell)$ with the maps

$$\mathfrak{B}((V^{\underline{op}})^r) \longrightarrow \mathfrak{B}(V^{\underline{op}})^{\underline{o}} = \mathfrak{B}(V)^{\underline{op}\,\underline{o}}$$

in (3.3) and (2.4). This proves part a), and part b) can be checked easily. $\square$

5. Bi-Products Revisited

Let H be a Hopf algebra with antipode S and suppose $R \in {}^H_H \mathcal{Y}D$ is a bialgebra in the category. The biproduct $R\#H$ of R and H is a bialgebra over k described as follows. As a vector space $R\#H = R \otimes H$ and $r\#h$ stands for the tensor $r \otimes h$. As a bialgebra $R\#H$ has the smash product and smash coproduct structures. Thus $1_{R\#H} = 1_R\#1_H$,

$$(r\#h)(r'\#h') = r(h_{(1)} \cdot r')\#h_{(2)}h'$$

for all $r, r' \in R$ and $h, h' \in H$,

$$\Delta(r\#h) = (r^{(1)}\#r^{(2)}{}_{(-1)}h_{(1)}) \otimes (r^{(2)}{}_{(0)}\#h_{(2)}), \quad \text{and} \quad \epsilon(r\#h) = \epsilon(r)\epsilon(h)$$

for all $r \in R$ and $h \in H$.

The map $j : H \longrightarrow R\#H$ defined by $j(h) = 1\#h$ for $h \in H$ is a bialgebra map and the map $\pi : R\#H \longrightarrow H$ defined by $\pi(r) = r\#1$ for $r \in R$ is an algebra map which satisfy $\pi \circ j = I_H$. Starting with the bialgebras $A = R\#H$, H and the maps j, π one can recover $R = R\#1$ as a bialgebra in ${}^H_H\mathcal{Y}D$. Consider the convolution product

$$(5.1) \qquad \Pi = I_A * (j \circ S \circ \pi)$$

which as an endomorphism of A given by $\Pi(a) = a_{(1)}j(S(\pi(a_{(2)})))$ for all $a \in A$. Observe that $\Pi(r\#h) = (r\#1)\epsilon(h)$ for all $r \in R$ and $h \in H$. In particular $R = \operatorname{Im}\Pi$. As a k-algebra R is merely a subalgebra of A. As a k-coalgebra

$$(5.2) \qquad \Delta_R(r) = \Pi(r_{(1)}) \otimes r_{(2)} \qquad \text{and} \qquad \epsilon_R(r) = \epsilon(r)$$

for all $r \in R$. As an object of ${}^H_H\mathcal{Y}D$ the left H-module action on R is given by

$$(5.3) \qquad h \cdot r = j(h_{(1)})rj(S(h_{(2)}))$$

for all $h \in H$ and $r \in R$ and as a left H-comodule action is given by

$$(5.4) \qquad \delta(r) = \pi(r_{(1)}) \otimes r_{(2)}$$

for all $r \in R$.

We are in now in a position to look at biproducts in more abstract terms. Let A be a bialgebra over k and suppose that $j : H \longrightarrow A$, $\pi : A \longrightarrow H$ are bialgebra maps which satisfy $\pi \circ j = I_H$. Let $\Pi : A \longrightarrow A$ be defined by (5.1) and set $R = \operatorname{Im}\Pi$. The mapping Π has many important properties which are basic for what follows and which we use without particular reference in Section 8; see [9] for example. First of all $\Pi \circ \Pi = \Pi$ and $\pi \circ \Pi = \eta_H \circ \epsilon_A$. Since $\Delta(\Pi(a)) =$

$a_{(1)}(j{\circ}S{\circ}\pi)(a_{(3)}){\otimes}\Pi(a_{(2)})$ for all $a \in A$ it now follows that $\Delta(R) \subseteq R{\otimes}A$ and

$$R = \operatorname{Im}\Pi = \{a \in A \,|\, a_{(1)}{\otimes}\pi(a_{(2)}) = a{\otimes}1\} = A^{co\,\pi}.$$

The last equation is definition. In particular R is a subalgebra of A. Since $\Delta(R) \subseteq A{\otimes}R$ then map $\Delta_R : A \longrightarrow A{\otimes}A$ defined by (5.2) satisfies $\Delta_R(R) \subseteq R{\otimes}R$. Using the fact that $\Pi(aj(h)) = \Pi(a)\epsilon(h)$ for all $a \in A$ and $h \in H$ it follows by direct calculation that $(R, \Delta_R, \epsilon|_R)$ is a k-coalgebra.

Note that $(A, \cdot_j, \delta_\pi) \in {}^H_H\mathcal{YD}$, where $h{\cdot}_j a = j(h_{(1)})aj(S(h_{(2)}))$ for all $h \in H$ and $a \in A$ and $\delta_\pi(a) = \pi(a_{(1)}){\otimes}a_{(2)}$ for all $a \in A$. Since $\Delta(R) \subseteq A{\otimes}R$ it follows that R is a left H-subcomodule of (A, δ_π). Since $h{\cdot}_j\Pi(a) = \Pi(j(h)a)$ for all $h \in H$ and $a \in A$ we see that R is a left H-submodule of $(A, \cdot_j)$. Therefore R is a subobject of $(A, \cdot_j, \delta_\pi)$ and the actions are those described in (5.3) and (5.4). In fact R with these structures is a bialgebra in ${}^H_H\mathcal{YD}$ and the map $R\#H \longrightarrow A$ determined by $r\#h \mapsto rj(h)$ is an isomorphism of k-bialgebras which we call the canonical isomorphism. We refer to R with these structures as the bialgebra in ${}^H_H\mathcal{YD}$ associated to (A, H, j, π).

The preceding discussion has been based on a bialgebra A over k with bialgebra maps $j : H \longrightarrow A$ and $\pi : A \longrightarrow H$ satisfying $\pi{\circ}j = I_H$. We have the same context for A^{op}, A^{cop}, thus for $A^{op\,cop}$, and A^o too. For $j : H^{op} \longrightarrow A^{op}$ and $\pi : A^{op} \longrightarrow H^{op}$, as well as $j : H^{cop} \longrightarrow A^{cop}$ and $\pi : A^{cop} \longrightarrow H^{cop}$, are bialgebra maps which satisfy $\pi{\circ}j = I_H$, and $\pi^o : H^o \longrightarrow A^o$ and $j^o : A^o \longrightarrow H^o$ are bialgebra maps which satisfy $j^o{\circ}\pi^o = (\pi{\circ}j)^o = I_{H^o}$. It will be important to us to understand A^{op} and A^o as biproducts. The analysis is rather detailed and will be carried out in Section 8. We do not need to deal with A^{cop}.

We now turn our attention to maps of biproducts. The result we need follows directly from definitions.

Proposition 5.1. *Let H, $\overline{H}$ be Hopf algebras over the field k and let $R \in {}^H_H\mathcal{YD}$, $\overline{R} \in {}^{\overline{H}}_{\overline{H}}\mathcal{YD}$ be bialgebras in their respective categories. Suppose $\varphi : H \longrightarrow \overline{H}$ is a bialgebra map and $\psi : R \longrightarrow \overline{R}$ is a map of k-algebras and coalgebras which is also φ-linear and colinear. Then the linear map $\psi\#\varphi : R\#H \longrightarrow \overline{R}\#\overline{H}$ defined by $(\psi\#\varphi)(r\#h) = \psi(r)\#\varphi(h)$ for all $r \in R$ and $h \in H$ is a map of bialgebras over k.* $\square$

Continuing with the statement of the proposition, note that $(\psi\#\varphi){\circ}j = \overline{j}{\circ}(\psi\#\varphi)$ and $\overline{\pi}{\circ}(\psi\#\varphi) = (\psi\#\varphi){\circ}\overline{\pi}$. Suppose that A, $\overline{A}$ are bialgebras over k. Let $j : H \longrightarrow A$, $\pi : A \longrightarrow H$ and $\overline{j} : \overline{H} \longrightarrow \overline{A}$, $\overline{\pi} : \overline{A} \longrightarrow \overline{H}$ be bialgebra maps which satisfy $\pi{\circ}j = I_A$, $\overline{\pi}{\circ}\overline{j} = I_{\overline{A}}$ respectively.

In light of the proposition a natural requirement for bialgebra maps $f : A \longrightarrow \overline{A}$ is $\overline{\pi} \circ f = f \circ \pi$ and $f \circ j = \overline{j} \circ f$. When this is the case $f(\mathrm{Im}\, j) \subseteq \mathrm{Im}\, \overline{j}$ and $\varphi : H \longrightarrow \overline{H}$ determined by $\overline{j} \circ \varphi = f \circ j$ is a bialgebra map, $f(A^{co\,\pi}) \subseteq \overline{A}^{co\,\overline{\pi}}$, and the restriction $f_r = f|_R$ is a map $f_r : R \longrightarrow \overline{R}$ of k-algebras, k-coalgebras, and is φ-linear and colinear. Furthermore the diagram

$$
\begin{array}{ccc}
R\#H & \xrightarrow{\;f_r\#\varphi\;} & \overline{R}\#\overline{H} \\[2mm]
\big\downarrow & & \big\downarrow \\[2mm]
A & \xrightarrow{\quad f \quad} & \overline{A}
\end{array}
$$

commutes, where the vertical maps are the k-bialgebra isomorphisms determined by $r\#h \mapsto rj(h)$ and $\overline{r}\#\overline{h} \mapsto \overline{r}\,\overline{j}(\overline{h})$ respectively.

6. $(R\#H)^{op}$ AS A BI-PRODUCT

Throughout this section H is a Hopf algebra with bijective antipode S, A is a bialgebra over k, and $j : H \longrightarrow A$, $\pi : A \longrightarrow H$ are bialgebra maps which satisfy $\pi \circ j = I_H$. We will use the results of Section 5 rather freely and in most cases without particular reference. As we noted in Section 1.2 the multiplicative opposite of a bi-product is a bi-product.

Let $R = A^{co\,\pi}$, let $(R, \cdot, \delta)$ be the structure of R as an object of $^{H}_{H}\mathcal{YD}$, and let $(R, m, \eta, \Delta, \epsilon)$ the bialgebra in $^{H}_{H}\mathcal{YD}$ associated to (A, H, j, π). We recall that $h\cdot r = j(h_{(1)})rj(S(h_{(2)}))$ for all $h \in H$ and $r \in R$ by (5.3), and $\delta(r) = \pi(r_{(1)})\otimes r_{(2)}$ for all $r \in R$ by (5.4).

As noted in Section 5 the maps $j : H^{op} \longrightarrow A^{op}$, $\pi : A^{op} \longrightarrow H^{op}$ are bialgebra maps which satisfy $\pi \circ j = I_{H^{op}}$. We first observe that $R = A^{co\,\pi} = (A^{op})^{co\,\pi}$. Let $(R, \cdot', \delta')$ be the structure of $R = (A^{op})^{co\,\pi}$ as an object in the category $^{H^{op}}_{H^{op}}\mathcal{YD}$ and let $(R, m', \eta', \Delta', \epsilon')$ be the bialgebra in the category associated with (A^{op}, H^{op}, j, π). The calculation

$$
\begin{aligned}
h^{op}\cdot' r &= j(h^{op}{}_{(1)})^{op} r^{op} j(S^{op}(h^{op}{}_{(2)}))^{op} \\
&= j(S^{-1}(h_{(2)}))rj(h_{(1)}) \\
&= j(S^{-1}(h_{(2)}))rj(s(S^{-1}(h_{(1)}))) \\
&= j(S^{-1}(h)_{(1)})rj(s(S^{-1}(h)_{(2)})) \\
&= S^{-1}(h)\cdot r
\end{aligned}
$$

for all $h^{op} = h \in H^{op}$ and $r \in R$ shows that $\cdot' = \cdot_{op}$. Since $\delta' = \delta$ it follows that $(R, \cdot', \delta') = (R, \cdot_{op}, \delta)$. Thus $R = R^{\underline{op}}$ as an object of $^{H^{op}}_{H^{op}}\mathcal{YD}$.

It is clear that $m' = m^{op}$, $\eta' = \eta$, and $\epsilon' = \epsilon$. To calculate Δ' we work from the definition $\Pi^{op} = I_{A^{op}} *^{op} (j \circ S^{op} \circ \pi)$ and compute $\Pi^{op}(a) = ((j \circ S^{-1} \circ \pi)(a_{(2)}))a_{(1)}$ for all $a = a^{op} \in A^{op}$. Now $\Pi(R) \subseteq A \otimes R$ and Π acts as the identity on R, which thus hold for Π^{op} as well. Since $\Pi(j(h)a) = h \cdot_j \Pi(a)$ for all $h \in H$ and $a \in A$, the calculation

$$
\begin{aligned}
\Delta'(r) &= \Pi^{op}(r^{op}{}_{(1)}) \otimes r^{op}{}_{(2)} \\
&= ((j \circ S^{-1} \circ \pi)(r_{(2)}))r_{(1)} \otimes r_{(3)} \\
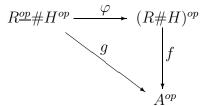&= \Pi(j(S^{-1}(\pi(r_{(2)})))r_{(1)}) \otimes r_{(3)} \\
&= S^{-1}(\pi(r_{(2)})) \cdot_j \Pi(r_{(1)}) \otimes r_{(3)} \\
&= S^{-1}(r^{(2)}{}_{(-1)}) \cdot r^{(1)} \otimes r^{(2)}{}_{(0)}
\end{aligned}
$$

for all $r \in R$ shows that $\Delta' = \Delta^{\underline{op}}$. We have shown that the bialgebra in $^{H^{op}}_{H^{op}}\mathcal{Y}D$ which is associated to (A^{op}, H^{op}, j, π) is $R^{\underline{op}}$.

Proposition 6.1. *Let H be an Hopf algebra with bijective antipode over k, let A be a bialgebra over k, and suppose that $j : H \longrightarrow A$, $\pi : A \longrightarrow H$ are bialgebra maps which satisfy $\pi \circ j = I_H$. Let $R = A^{co\,\pi}$ and let $(R, m, \eta, \Delta, \epsilon)$ be the bialgebra in $^H_H\mathcal{Y}D$ associated to (A, H, j, π). Then:*

  a) *$j : H^{op} \longrightarrow A^{op}$ and $\pi : A^{op} \longrightarrow H^{op}$ are bialgebra maps which satisfy $\pi \circ j = I_{H^{op}}$, $R^{\underline{op}} = R$ as a vector space, and the bialgebra in $^{H^{op}}_{H^{op}}\mathcal{Y}D$ associated to (A^{op}, H^{op}, j, π) is $(R^{\underline{op}}, m^{\underline{op}}, \eta, \Delta^{\underline{op}}, \epsilon)$.*
  b) *The map $\varphi : R^{\underline{op}} \# H^{op} \longrightarrow (R \# H)^{op}$ given by $\varphi(r \# h) = (1 \# h)(r \# 1)$ for all $r \in R$ and $h \in H$ is an isomorphism of bialgebras. Furthermore the diagram*

$$
\begin{array}{ccc}
R^{\underline{op}} \# H^{op} & \xrightarrow{\ \varphi\ } & (R \# H)^{op} \\
& \searrow{\scriptstyle g} & \big\downarrow{\scriptstyle f} \\
& & A^{op}
\end{array}
$$

  commutes, where $f : R \# H \longrightarrow A$ and $g : R^{\underline{op}} \# H^{op} \longrightarrow A^{op}$ are the canonical isomorphisms.

PROOF: We have established part a). As for part b), we first note that $f(r \# h) = rj(h)$ and $g(r \# h) = r^{op}j(h)^{op} = j(h)r$ for all $r \in R$ and $h \in H$. Therefore $f \circ \varphi = g$. This means the diagram commutes and $\varphi = f^{-1} \circ g$ is an isomorphism of bialgebras. $\square$

Observe that $\varphi^{-1}(r \# h) = h_{(1)} \cdot_{op} r \# h_{(2)} = S^{-1}(h_{(1)}) \cdot r \# h_{(2)}$ for all $r \in R$ and $h \in H$.

7. $(R\#H)^o$ as a Bi-Product

As in the preceding section, H is a Hopf algebra with bijective antipode S, A is a bialgebra over k, and $j : H \longrightarrow A$, $\pi : A \longrightarrow H$ are bialgebra maps which satisfy $\pi \circ j = I_H$. Again we will use the results of Section 5 rather freely and in most cases without particular reference. We observed in Section 1.2 that the dual of a bi-product is a bi-product.

As noted in Section 5 the maps $\pi^o : H^o \longrightarrow A^o$, $j^o : A^o \longrightarrow H^o$ are bialgebra maps which satisfy $j^o \circ \pi^o = I_{H^o}$. We will show that $(A^o)^{co\,j^o}$ can be identified with $R^{\underline{o}}$, find the structure of $R^{\underline{o}}$ as an object of $^{H^o}_{H^o}\mathcal{YD}$, and then find its structure as the bialgebra in $^{H^o}_{H^o}\mathcal{YD}$ associated to (A^o, H^o, π^o, j^o).

Let $R' = (A^o)^{co\,j^o}$ and $a^o \in A^o$. Then $a^o \in R'$ if and only if $a^o_{(1)} \otimes j^o(a^o_{(2)}) = a^o \otimes \epsilon$, or equivalently $a^o(aj(h)) = a^o(a)\epsilon(h)$ for all $a \in A$ and $h \in H$. Recall from Section 5 that the map $R\#H \longrightarrow A$ determined by $r\#h \mapsto rj(h)$ for all $r \in R$ and $h \in H$ is an isomorphism of bialgebras. Since $A = Rj(H)$ it follows that $a^o \in R'$ if and only if $a^o(rj(h)) = a^o(r)\epsilon(h)$ for all $r \in R$ and $h \in H$. The isomorphism gives rise to a linear embedding $i : R^* \longrightarrow A^*$, where $i(r^*)(rj(h)) = r^*(r)\epsilon(h)$ for all $r^* \in R^*$, $r \in R$, and $h \in H$. Observe that $R' \subseteq \operatorname{Im} i$. Thus we can understand R' in terms of R^* via the embedding.

Our first claim is that $i(R^{\underline{o}}) = R'$. A consequence of the claim is that the restriction

$$(7.1) \qquad\qquad i|_{R^{\underline{o}}} : R^{\underline{o}} \longrightarrow R'$$

is a linear isomorphism.

To prove our claim, first of all suppose that $r^{\underline{o}} \in R^{\underline{o}}$. To show that $i(r^{\underline{o}}) \in R'$ we need only show that $i(r^{\underline{o}}) \in A^o$. Since $r^{\underline{o}} \in R^{\underline{o}}$, by definition $r^{\underline{o}}(J) = (0)$ for some cofinite subspace J of R which is an ideal and a left H-submodule of R. We will use the commutation relations

$$j(h)r = (h_{(1)} \cdot_j r)j(h_{(2)}) \qquad \text{and} \qquad rj(s(h)) = j(s(h_{(1)}))(h_{(2)} \cdot_j r)$$

for all $h \in H$ and $r \in R$. Since S is onto and J is a left H-subcomodule of R it follows from the commutations relations that $j(H)J = Jj(H)$. Thus $Jj(H)$ is a left ideal of A. Now $i(r^{\underline{o}})$ vanishes on $Jj(H)$ and $Rj(H)^+ = R(j(H) \cap \operatorname{Ker} \epsilon)$ as well. Since $H \cdot_j R \subseteq R$ and $j(H)^+$ is a left ideal of $j(H)$, by the first commutation relation $Rj(H)^+$ is a left ideal of A. Since $Jj(H) + Rj(H)^+$ is a cofinite left ideal of A on which $i(r^{\underline{o}})$ vanishes, it follows that $i(r^{\underline{o}})$ vanishes on a cofinite ideal of A. Thus $i(r^{\underline{o}}) \in A^o$ as required.

Now suppose that $a^o \in R'$. Since $R' \subseteq \operatorname{Im} i$ it follows that $i(r^*) = a^o$ for some $r^* \in R^*$. By definition $a^o(I) = (0)$ for some cofinite ideal I of A. Since ideals of A are also left H-submodules of A, and the subalgebra R of A is also a left H-submodule, $J = R \cap I$ is a cofinite ideal of R which is also a left H-submodule of R. As $(0) = a^o(J) = i(r^*)(J) = r^*(J)$ we conclude that $r^* \in R^{\underline{o}}$. We have completed the proof of the claim.

Thus $R^{\underline{o}}$ and R' can be identified as vector spaces by the map of (7.1). Accordingly we will think of R' as $R^{\underline{o}}$ and show that R' is $R^{\underline{o}}$ as an object of ${}^{H^o}_{H^o}\mathcal{YD}$ and R' is $R^{\underline{o}}$ as the bialgebra in the category associated to (A^o, H^o, π^o, j^o).

Let $h^o \in H^o$ and $r^{\underline{o}} \in R^{\underline{o}}$. The left H^o-module structure on R' is given by $h^o \cdot_{\pi^o} i(r^{\underline{o}}) = \pi^o(h^o{}_{(1)}) i(r^{\underline{o}}) \pi^o(S^o(h^o{}_{(2)}))$. We evaluate both sides of this equation at $r \in R$. Since $r_{(1)} \otimes \pi(r_{(2)}) = r \otimes 1$ we have $\pi(r_{(1)}) \otimes r_{(2)} \otimes \pi(r_{(3)}) = \pi(r_{(1)}) \otimes r_{(2)} \otimes 1$ from which $\pi(r_{(1)}) s(\pi(r_{(3)})) \otimes r_{(2)} = \pi(r_{(1)}) \otimes r_{(2)} = r_{(-1)} \otimes r_{(0)}$ follows. Thus

$$
\begin{aligned}
\left(h^o \cdot_{\pi^o} i(r^{\underline{o}})\right)(r) &= \left(\pi^o(h^o{}_{(1)})(r_{(1)})\right)\left(i(r^{\underline{o}})(r_{(2)})\right)\left(\pi^o(S^o(h^o{}_{(2)}))(r_{(3)})\right) \\
&= \left(h^o{}_{(1)}(\pi(r_{(1)}))\right)\left(i(r^{\underline{o}})(r_{(2)})\right)\left(h^o{}_{(2)}(S(\pi(r_{(3)})))\right) \\
&= h^o\left(\pi(r_{(1)})S(\pi(r_{(3)}))\right) i(r^{\underline{o}})(r_{(2)}) \\
&= h^o(r_{(-1)}) r^{\underline{o}}(r_{(0)}).
\end{aligned}
$$

We have shown that $\left(h^o \cdot_{\pi^o} i(r^{\underline{o}})\right)(r) = h^o(r_{(-1)}) r^{\underline{o}}(r_{(0)})$ for all $h^o \in H^o$, $r^{\underline{o}} \in R^{\underline{o}}$, and $r \in R$.

The left H^o-comodule structure on R' is the subcomodule structure afforded by (A^o, δ_{j^o}). Now $\delta_{j^o}(i(r^{\underline{o}})) = j^o(i(r^{\underline{o}})_{(1)}) \otimes i(r^{\underline{o}})_{(2)}$. Using the first commutation relation above we calculate

$$
\begin{aligned}
\delta_{j^o}(i(r^{\underline{o}}))(h \otimes r) &= i(r^{\underline{o}})_{(1)}(j(h)) i(r^{\underline{o}})_{(2)}(r) \\
&= i(r^{\underline{o}})(j(h)r) \\
&= i(r^{\underline{o}})((h_{(1)} \cdot_j r) j(h_{(2)})) \\
&= r^{\underline{o}}(h_{(1)} \cdot_j r) \epsilon(h_{(2)}) \\
&= r^{\underline{o}}(h \cdot_j r)
\end{aligned}
$$

for all $h \in H$ and $r \in R$. We have shown that $R' = R^{\underline{o}}$ as an object in ${}^{H^o}_{H^o}\mathcal{YD}$.

Next we consider the product in R'. Let $r^{\underline{o}}, r'^{\underline{o}} \in R^{\underline{o}}$. Since $A = Rj(H)$ it is easy to see that $i(r^{\underline{o}})$ is determined on R. Thus for $r \in R$

the calculation

$$
\begin{aligned}
i(r^{\underline{o}})(r^{(1)})i(r'^{\underline{o}})(r^{(2)}) &= i(r^{\underline{o}})(\Pi(r_{(1)}))i(r'^{\underline{o}})(r_{(2)}) \\
&= i(r^{\underline{o}})\left(r_{(1)}j((s\circ\pi)(r_{(2)}))\right)i(r'^{\underline{o}})(r_{(3)}) \\
&= i(r^{\underline{o}})(r_{(1)})\epsilon((s\circ\pi)(r_{(2)}))i(r'^{\underline{o}})(r_{(3)}) \\
&= i(r^{\underline{o}})(r_{(1)})i(r'^{\underline{o}})(r_{(2)}) \\
&= (i(r^{\underline{o}})i(r'^{\underline{o}}))(r)
\end{aligned}
$$

shows that the product $r^{\underline{o}}r'^{\underline{o}}$ is derived from dual algebra of $(R^{\underline{o}}, \Delta^{\underline{o}})$.

Finally we consider the coproduct of R'. First we calculate Π_{A^o} in terms of $\Pi_A = \Pi$. Let $p \in A^o$ and $a \in A$. Then

$$
\begin{aligned}
(\Pi_{A^o}(p))(a) &= \left(p_{(1)}((\pi^o\circ S^o\circ j^o)(p_{(2)}))\right)(a) \\
&= p_{(1)}(a_{(1)})p_{(2)}((j\circ s\circ\pi)(a_{(2)})) \\
&= p(a_{(1)}((j\circ s\circ\pi)(a_{(2)}))) \\
&= p(\Pi(a))
\end{aligned}
$$

implies $\Pi_{A^o} = (\Pi_A)^o$. Let $r^{\underline{o}} \in R^{\underline{o}}$. By definition

$$
\Delta_{R'}(i(r^{\underline{o}})) = \Pi_{A^o}(i(r^{\underline{o}})_{(1)})\otimes i(r^{\underline{o}})_{(2)}.
$$

Thus for $r, r' \in R$ we compute

$$
\begin{aligned}
\Delta_{R'}(i(r^{\underline{o}}))(r\otimes r') &= \left(\Pi_{A^o}(i(r^{\underline{o}})_{(1)})(r)\right)\left(i(r^{\underline{o}})_{(2)}(r')\right) \\
&= \left(i(r^{\underline{o}})_{(1)}(\Pi(r))\right)\left(i(r^{\underline{o}})_{(2)}(r')\right) \\
&= i(r^{\underline{o}})(\Pi(r)r') \\
&= i(r^{\underline{o}})(rr')
\end{aligned}
$$

since Π acts as the identity on R. Thus the coproduct for $R' = R^{\underline{o}}$ is that of the dual coalgebra arising from the subalgebra R of A. Therefore the bialgebra R' in the category ${}^{H^o}_{H^o}\mathcal{YD}$ associated to (A^o, H^o, π^o, j^o) is $R^{\underline{o}}$. We regard $R^*\otimes H^*$ as a subspace of the vector space $(R\#H)^* = (R\otimes H)^*$ in the natural way.

Proposition 7.1. *Let H be an Hopf algebra with bijective antipode over k, let A be a bialgebra over k and suppose that $j : H \longrightarrow A$, $\pi : A \longrightarrow H$ are bialgebra maps which satisfy $\pi\circ j = I_H$. Let $R = A^{co\,\pi}$ and let $(R, m, \eta, \Delta, \epsilon)$ be the bialgebra in ${}^H_H\mathcal{YD}$ associated to (A, H, j, π). Then:*

    a) *$\pi^o : H^o \longrightarrow A^o$ and $j^o : A^o \longrightarrow H^o$ are bialgebra maps which satisfy $j^o\circ\pi^o = I_{H^o}$, $R^{\underline{o}} = A^{o\,co\,j^o}$ as a vector space under the identification $r^{\underline{o}}(rj(h)) = r^{\underline{o}}(r)\epsilon(h)$ for all $r \in R$ and $h \in H$, and $(R^{\underline{o}}, \Delta^{\underline{o}}, \epsilon, m^{\underline{o}}, \eta)$ is the bialgebra in ${}^{H^o}_{H^o}\mathcal{YD}$ associated to (A^o, H^o, π^o, j^o).*

b) *The map $\vartheta : R^{\underline{o}}\#H^o \longrightarrow (R\#H)^o$ given by $\vartheta(r^{\underline{o}}\#h^o) = r^{\underline{o}}\otimes h^o$*
for all $r^{\underline{o}} \in R^{\underline{o}}$ and $h^o \in H^o$ is an isomorphism of bialgebras.
Furthermore the diagram

$$
\begin{array}{ccc}
R^{\underline{o}}\#H^o & \xrightarrow{\ \vartheta\ } & (R\#H)^o \\
{\scriptstyle i|_{R^{\underline{o}}}\#I_{H^o}}\downarrow & & \downarrow{\scriptstyle (f^{-1})^o} \\
R'\#H^o & \xrightarrow{\ g\ } & A^o
\end{array}
$$

commutes, where $R' = A^{o\,co\,j^o}$, $i|_{R^{\underline{o}}}$ is the linear isomorphism
of (7.1), and the maps $f : R\#H \longrightarrow A$, $g : R'\#H^o \longrightarrow A^o$ are
the canonical isomorphisms.

PROOF: We have established part a). As for part b), we first note that
the structures on $R^{\underline{o}}$ are due to the identification $i|_{R^{\underline{o}}} : R^{\underline{o}} \longrightarrow R'$. Thus
$i|_{R^{\underline{o}}}\#I_{H^o}$ is an isomorphism of bialgebras by Proposition 5.1. Now g
and $(f^{-1})^o$ are bialgebra isomorphisms also. Thus part b) will follow
once we show that the diagram commutes.

Let $r^{\underline{o}} \in R^{\underline{o}}$, $h^o \in H^o$, $r \in R$, $h \in H$, and set $a = rj(h)$. Then

$$
\begin{aligned}
\left(\left((f^{-1})^o\circ\vartheta\right)(r^{\underline{o}}\#h^o)\right)(a) &= \vartheta(r^{\underline{o}}\#h^o)(f^{-1}(a)) \\
&= (r^{\underline{o}}\otimes h^o)(r\#h) \\
&= r^{\underline{o}}(r)h^o(h).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\left(\left(g\circ(i|_{R^{\underline{o}}}\#I_{H^o})\right)(r^{\underline{o}}\#h^o)\right)(a) &= \left(g(i(r^{\underline{o}})\#h^o)\right)(a) \\
&= \left(i(r^{\underline{o}})\pi^o(h^o)\right)(rj(h)) \\
&= i(r^{\underline{o}})(r_{(1)}j(h_{(1)}))\pi^o(h^o)(r_{(2)}j(h_{(2)})) \\
&= i(r^{\underline{o}})(r_{(1)}j(h_{(1)}))h^o(\pi(r_{(2)}j(h_{(2)}))) \\
&= r^{\underline{o}}(r_{(1)})\epsilon(h_{(1)})h^o(\pi(r_{(2)})(\pi\circ j)(h_{(2)})) \\
&= r^{\underline{o}}(r_{(1)})h^o(\pi(r_{(2)})h)) \\
&= r^{\underline{o}}(r)h^o(1h)) \\
&= r^{\underline{o}}(r)h^o(h).
\end{aligned}
$$

Since $A = Rj(H)$ our calculations show that the diagram of part b)
commutes. $\square$

8. BIALGEBRAS $\mathcal{H} = (\mathcal{U}\otimes\mathcal{A})^\sigma$ WHEN $\mathcal{U}, \mathcal{A}$ ARE BI-PRODUCTS

Let K and H be Hopf algebras with bijective antipodes over the field
k and suppose that $U = T\#K$, $A = R\#H$ are bi-products. We de-
scribe an extensive class of linear forms $(T\#K)\otimes(R\#H) \longrightarrow k$ which

satisfy (A.1)–(A.4). All such forms are in bijective correspondence with the bialgebra maps $T\#K \longrightarrow (R\#H)^{o\,cop} = (R\#H)^{op\,o}$. Our forms are derived from certain bialgebra maps and determine two-cocycles σ.

8.1. **The Linear Form** $\beta\#\tau : (T\#K)\otimes(R\#H) \longrightarrow k$**.** We use the isomorphisms of the two preceding sections to construct bialgebra maps

$$T\#K \longrightarrow (R\#H)^{op\,o} \cong R^{\underline{op}\,\underline{o}}\#H^{op\,o}$$

which determine linear forms $\beta\#\tau : (T\#K)\otimes(R\#H) \longrightarrow k$ satisfying (A.1)–(A.4). In the subsequent section we will investigate the special case when K and H are group algebras of abelian groups. These are fundamental, interesting in their own right, and arise in representations theory of pointed Hopf algebras [3, 10].

Theorem 8.1. *Let K and H be Hopf algebras with bijective antipodes over the field k. Suppose that T and R are bialgebras in the categories $^K_K\mathcal{Y}D$ and $^H_H\mathcal{Y}D$ respectively and that $\tau : K\otimes H \longrightarrow k$ and $\beta : T\otimes R \longrightarrow k$ are linear forms, where τ satisfies (A.1)–(A.4), β satisfies (B.1)–(B.4), and τ and β satisfy (C.1)–(C.2). Let*

$$\beta\#\tau : (T\#K)\otimes(R\#H) \longrightarrow k$$

be the linear form determined by

$$(\beta\#\tau)(t\#k, r\#h) = \beta(t, S^{-1}(h_{(1)})\cdot r)\tau(k, h_{(2)})$$

for all $t \in T$, $k \in K$, $r \in R$, and $h \in H$. Then:

a) $(\beta\#\tau)_\ell$ *is the composite*

$$T\#K \xrightarrow{\beta_\ell\#\tau_\ell} R^{\underline{op}\,\underline{o}}\#H^{op\,o} \xrightarrow{\vartheta} (R^{\underline{op}}\#H^{op})^o \xrightarrow{(\varphi^{-1})^o} (R\#H)^{op\,o},$$

   where ϑ and φ are defined in part b) of Propositions 7.1 and 6.1 respectively. Thus $\beta\#\tau$ satisfies (A.1)–(A.4).

b) $(\beta\#\tau)_\ell = (\varphi^{-1})^*\circ(\beta_\ell\otimes\tau_\ell)$ *and* $(\beta\#\tau)_r = (\beta_r\otimes\tau_r)\circ\varphi^{-1}$.

c) $\beta\#\tau$ *is left (respectively right) non-singular if and only if β and τ are left (respectively right) non-singular.*

PROOF: Since τ satisfies (A.1)–(A.4), its equivalent (1.1), which is $\mathrm{Im}\,\tau_\ell \subseteq H^{op\,o}$ and $\tau_\ell : K \longrightarrow H^{op\,o}$ is a bialgebra map, holds. Likewise, since β satisfies (B.1)–(B.4), by Lemma 4.1 it follows that $\mathrm{Im}\,\beta_\ell \subseteq R^{\underline{op}\,\underline{o}}$ and $\beta_\ell : T \longrightarrow R^{\underline{op}\,\underline{o}}$ is a map of algebras and a map of coalgebras. Now β_ℓ is τ_ℓ-linear and colinear since τ and β satisfy (C.1)–(C.2) by Proposition 4.2. Therefore $\beta_\ell\#\tau_\ell : T\#K \longrightarrow (R^{\underline{op}})^{\underline{o}}\#(H^{op})^o$ is a bialgebra map by Proposition 5.1. At this point part a) follows by Propositions 6.1 and 7.1.

Part b) is a direct consequence of definitions. As for part c) we first note that the tensor product of two linear maps is one-one if and only if each tensorand is. Since φ^{-1} and $(\varphi^{-1})^*$ are linear isomorphisms, part c) now follows from part b). $\square$

Apropos of the theorem, requiring that $\beta\#\tau$ be left or right non-singular seems to be a rather stringent condition. We will find it very natural, and desirable, for β to be non-singular in connection with representation theory. See Section 9.

We shall call a tuple $(K, H, \tau, T, R, \beta)$ which satisfies the hypothesis of the preceding theorem 2-*cocycle twist datum*. In applications morphisms of algebras of the type $((T\#K)\otimes(R\#H))^\sigma$ will be of interest to us. As an immediate consequence of Theorem 8.1, Propositions 5.1, and [10, Proposition 4.3]:

Corollary 8.2. *Let K and H be Hopf algebras with bijective antipodes over the field k and let $(K, H, \tau, T, R, \beta)$ and $(\overline{K}, \overline{H}, \overline{\tau}, \overline{T}, \overline{R}, \overline{\beta})$ be two-cocycle twist data. Suppose that $\varphi : K \longrightarrow \overline{K}$ and $\nu : H \longrightarrow \overline{H}$ are bialgebra maps, $f : T \longrightarrow \overline{T}$ and $g : R \longrightarrow \overline{R}$ are algebra and coalgebra maps, where f is φ-linear and colinear and g is ν-linear and colinear. Assume further that $\overline{\tau}\circ(\varphi\otimes\nu) = \tau$ and $\overline{\beta}\circ(f\otimes g) = \beta$. Then $(\overline{\beta}\#\overline{\tau})\circ((f\#g)\otimes(\varphi\#\nu)) = \beta\#\tau$. In particular*

$$(f\#\varphi)\otimes(g\#\nu) : \big((T\#K)\otimes(R\#H)\big)^\sigma \longrightarrow \big((\overline{T}\#\overline{K})\otimes(\overline{R}\#\overline{H})\big)^{\overline{\sigma}}$$

is a bialgebra map. $\square$

8.2. **The Fundamental Case $U = \mathfrak{B}(W)\#K$ and $A = \mathfrak{B}(V)\#H$.** We specialize the results of the preceding section to the case of most interest to us: when $U = \mathfrak{B}(W)\#K$ and $A = \mathfrak{B}(V)\#H$ are biproducts of Nichols algebras with Hopf algebras having bijective antipodes. We are able to express assumptions involving $\mathfrak{B}(W)$ and $\mathfrak{B}(V)$ in terms of W and V.

Theorem 8.3. *Let K and H be Hopf algebras with bijective antipodes over the field k and let $\tau : K\otimes H \longrightarrow k$ be a linear form which satisfies (A.1)–(A.4). Suppose $W \in {}^K_K\mathcal{YD}$, $V \in {}^H_H\mathcal{YD}$, $\beta : W\otimes V \longrightarrow k$ is a linear form, and τ and β satisfy (C.1)–(C.2). Then:*

a) *$(K, H, \tau, \mathfrak{B}(W), \mathfrak{B}(V), \mathfrak{B}(\beta))$ is a two-cocycle twist datum.*
b) *The linear form $\mathfrak{B}(\beta)\#\tau : (\mathfrak{B}(W)\#K)\otimes(\mathfrak{B}(V)\#H) \longrightarrow k$ satisfies (A.1)–(A.4).*

PROOF: The form $\mathfrak{B}(\beta) : \mathfrak{B}(W)\otimes\mathfrak{B}(V) \longrightarrow k$ of Corollary 4.3 satisfies the hypothesis of Theorem 8.1. $\square$

We shall call a tuple $(K, H, \tau, W, V, \beta)$ which satisfies the hypothesis of the preceding theorem *Yetter—Drinfel'd 2-cocycle twist datum.* By the preceding theorem if $(K, H, \tau, W, V, \beta)$ is a Yetter–Drinfel'd two-cocycle twist datum then $(K, H, \tau, \mathfrak{B}(W), \mathfrak{B}(V), \mathfrak{B}(\beta))$ is a two-cocycle twist datum. We now turn our attention to morphisms.

Proposition 8.4. *Let K, H be Hopf algebras with bijective antipodes over k and $(K, H, \tau, W, V, \beta)$ and $(\overline{K}, \overline{H}, \overline{\tau}, \overline{W}, \overline{V}, \overline{\beta})$ be Yetter–Drinfel'd two-cocycle twist data. Suppose that $\varphi : K \longrightarrow \overline{K}$ and $\nu : H \longrightarrow \overline{H}$ are bialgebra maps which satisfy $\overline{\tau} \circ (\varphi \otimes \nu) = \tau$. Let $f : W \longrightarrow \overline{W}$ be φ-linear and colinear, let $g : V \longrightarrow \overline{V}$ be ν-linear and colinear, and suppose that $\overline{\beta} \circ (f \otimes g) = \beta$. Then*

$$F : \big((\mathfrak{B}(W)\#K) \otimes (\mathfrak{B}(V)\#H)\big)^{\sigma} \longrightarrow \big((\mathfrak{B}(\overline{W})\#\overline{K}) \otimes (\mathfrak{B}(\overline{V})\#\overline{H})\big)^{\overline{\sigma}}$$

is a bialgebra map, where $F = (\mathfrak{B}(f)\#\varphi) \otimes (\mathfrak{B}(g)\#\nu)$.

PROOF: First of all we observe that $(K, H, \tau, \mathfrak{B}(W), \mathfrak{B}(V), \mathfrak{B}(\beta))$ and $(\overline{K}, \overline{H}, \overline{\tau}, \mathfrak{B}(\overline{W}), \mathfrak{B}(\overline{V}), \mathfrak{B}(\overline{\beta}))$ are two-cocycle twist data by part a) of Theorem 8.3. By assumption $\overline{\tau} \circ (\varphi \otimes \nu) = \tau$. Since $\overline{\beta} \circ (f \otimes g) = \beta$, it follows by Corollary 4.3 that $\mathfrak{B}(\overline{\beta}) \circ (\mathfrak{B}(f) \otimes \mathfrak{B}(g)) = \mathfrak{B}(\beta)$. At this point we apply Corollary 8.2 to complete the proof. $\square$

Let $(K, H, \tau, W, V, \beta)$ be a Yetter-Drinfel'd two-cocycle twist datum. Axiom (C.1) implies that $V^{\perp}$ is a left K-submodule of W and axiom (C.2) implies that $W^{\perp}$ is a left H-submodule of V. Thus the projections $\pi_W : W \longrightarrow W/V^{\perp}$ and $\pi_V : V \longrightarrow V/W^{\perp}$ are module maps. If $V^{\perp}$ is a left K-subcomodule of W then π_W is also a left K-comodule map and likewise if $W^{\perp}$ is a left H-subcomodule of V then π_V is also a left H-comodule map.

Suppose that $V^{\perp}$ and $W^{\perp}$ are subcomodules. By part c) of Corollary 1.3 note that $\mathfrak{B}(\pi_W) : \mathfrak{B}(W) \longrightarrow \mathfrak{B}(W/V^{\perp})$ and $\mathfrak{B}(\pi_W) : \mathfrak{B}(W) \longrightarrow \mathfrak{B}(W/V^{\perp})$ are onto since π_W and π_V are. Recall that the linear form $\overline{\beta} : W/V^{\perp} \otimes V/W^{\perp} \longrightarrow k$ determined by $\overline{\beta} \circ (\pi_W \otimes \pi_V) = \beta$ is nonsingular. With φ, ν the identity, $f = \pi_W$, and $g = \pi_W$, the preceding proposition gives:

Corollary 8.5. *Let K, H be Hopf algebras with bijective antipodes over k and let $(K, H, \tau, W, V, \beta)$ be a Yetter–Drinfel'd two-cocycle twist datum. Suppose that $V^{\perp}$ and $W^{\perp}$ are subcomodules of W and V respectively. Let $\pi_W : W \longrightarrow W/V^{\perp}$ and $\pi_V : V \longrightarrow V/W^{\perp}$ be the projections. Then $(K, H, \tau, W/V^{\perp}, V/W^{\perp}, \overline{\beta})$ is a Yetter–Drinfel'd two-cocycle twist datum and*

$$\big((\mathfrak{B}(W)\#K) \otimes (\mathfrak{B}(V)\#H)\big)^{\sigma} \xrightarrow{F} \big((\mathfrak{B}(W/V^{\perp})\#K) \otimes (\mathfrak{B}(V/W^{\perp})\#H)\big)^{\overline{\sigma}}$$

is a surjective bialgebra map, where $F = (\mathfrak{B}(\pi_W)\#I_K)\otimes(\mathfrak{B}(\pi_V)\#I_H)$.
□

The preceding corollary can be used in our study of a class of irreducible representations of an extensive class of examples of two-cocycle twists in [10]. We will be able to replace the domain of F by its image and thus assume that β is non-singular. The non-singularity of β has very interesting consequences.

9. The Case when K and H are Group Algebras of Abelian Groups

Here we specialize the results of Section 8.2 to a very typical case. Let Γ be an abelian group. We set $_{k[\Gamma]}^{k[\Gamma]}\mathcal{Y}D = {}_\Gamma^\Gamma\mathcal{Y}\mathcal{D}$. Suppose that $V \in {}_\Gamma^\Gamma\mathcal{Y}\mathcal{D}$, $g \in \Gamma$, and $\chi \in \widehat{\Gamma}$. We set

$$V_g = \{v \in V \mid \delta(v) = g\otimes v\}$$

and

$$V_g^\chi = \{v \in V_g \mid h{\cdot}v = \chi(h)v \text{ for all } h \in \Gamma\}.$$

A Yetter-Drinfel'd module in $_\Gamma^\Gamma\mathcal{Y}\mathcal{D}$ can be described as a Γ-graded vector space which is a Γ-module such that all g-homogeneous components $g \in \Gamma$ are stable under the Γ-action.

In this section we fix abelian groups Λ and Γ, positive integers n and m, elements $z_1,\ldots,z_n \in \Lambda$ and $g_1,\ldots,g_m \in \Gamma$, and nontrivial characters $\eta_1,\ldots,\eta_n \in \widehat{\Lambda}$ and $\chi_1,\ldots,\chi_m \in \widehat{\Gamma}$. Suppose $W \in {}_\Lambda^\Lambda\mathcal{Y}\mathcal{D}$ has basis $u_i \in W_{z_i}^{\eta_i}, 1 \le i \le n$, and that $V \in {}_\Gamma^\Gamma\mathcal{Y}\mathcal{D}$ has basis $a_j \in V_{g_j}^{\chi_j}, 1 \le j \le m$.

The following corollaries will play a useful role in [10].

Corollary 9.1. *In addition to the above, let $\varphi : \Lambda \longrightarrow \widehat{\Gamma}$ be a group homomorphism, $s : \{1,\ldots,n\} \longrightarrow \{1,\ldots,m\}$ be a function, and $\lambda_1,\ldots,\lambda_n \in k$.*

Let $\tau : k[\Lambda] \otimes k[\Gamma] \longrightarrow k$ and $\beta : W\otimes V \longrightarrow k$ be the linear forms defined by

(9.1) $\tau(z \otimes g) = \varphi(z)(g)$ *and* $\beta(u_i\otimes a_j) = \lambda_i\delta_{s(i),j}$

for all $z \in \Lambda, g \in \Gamma, 1 \le i \le n$ and $1 \le j \le m$. Assume further that for all $1 \le i \le n$ with $\lambda_i \ne 0$ and $z \in \Lambda$

(9.2) $\varphi(z_i) = \chi_{s(i)}^{-1}$ *and* $\eta_i(z) = \varphi(z)(g_{s(i)}).$

Then $(k[\Lambda], k[\Gamma], \tau, W, V, \beta)$ is a Yetter–Drinfel'd two-cocycle twist datum, and the corresponding Hopf algebra map

$$\Phi = (\mathfrak{B}(\beta)\#\tau)_\ell : \mathfrak{B}(W)\#k[\Lambda] \longrightarrow (\mathfrak{B}(V)\#k[\Gamma])^{o\,cop}$$

can be described as follows.

For all $1 \leq i \leq n$ there are a unique algebra map

(9.3) $\gamma_i : \mathfrak{B}(V)\#k[\Gamma] \longrightarrow k$ *with* $\gamma_i(a_j\#1) = 0, \gamma_i(1\#g) = \varphi(z_i)(g)$

for all $1 \leq j \leq m$ and $g \in \Gamma$, and a unique (ε, γ_i)-derivation

(9.4) $\delta_i : \mathfrak{B}(V)\#k[\Gamma] \longrightarrow k$ *with* $\delta_i(a_j\#1) = \beta(u_i \otimes a_j), \delta_i(1\#g) = 0$

for all $1 \leq j \leq m$ and $g \in \Gamma$.

Then the algebra map $\Phi : \mathfrak{B}(W)\#k[\Lambda] \longrightarrow (\mathfrak{B}(V)\#k[\Gamma])^o$ *is determined by*

(9.5) $$\Phi(1\#z_i) = \gamma_i, \quad \Phi(u_i\#1) = \delta_i$$

for all $1 \leq i \leq n$.

PROOF: We first show that $(k[\Gamma], k[\Lambda], \tau, W, V, \beta)$ is Yetter–Drinfel'd two-cocycle datum. Since H is commutative τ satisfies (1.1), an equivalent of (A.1)–(A.4). Note that (C.2) holds for τ and β if and only if for all $1 \leq i \leq n$ and $1 \leq j \leq m$ the equation $\beta(u_i \leftharpoonup \tau_r(g), a_j) = \beta(u_i, S^{-1}(g) \cdot a_j)$ holds for all $g \in \Gamma$. The latter is $\tau_r(g)(z_i)\beta(u_i, a_j) = \beta(u_i, \chi_j(g^{-1})a_j)$, or $\tau_\ell(z_i)(g)\lambda_i\delta_{\sigma(i),j} = \chi_j(g^{-1})\lambda_i\delta_{\sigma(i),j}$, an equivalent of the first equation of (9.2). Next we note that (C.1) is equivalent to $\beta(z \cdot u_i, a_j) = \beta(u_i, a_j \leftharpoonup \tau_\ell(z))$ which is the same as $\beta(\eta_i(z)u_i, a_j) = \beta(u_i, \tau_\ell(z)(g_j)a_j)$, or $\eta_i(z)\lambda_i\delta_{\sigma(i),j} = \varphi(z)(g_j)\lambda_i\delta_{\sigma(i),j}$, for all $z \in \Lambda$, $1 \leq i \leq n$, and $1 \leq j \leq m$. The latter is equivalent to the second equation of (9.2).

We have shown that $(k[\Gamma], k[\Lambda], \tau, W, V, \beta)$ is Yetter–Drinfel'd two-cocycle datum, and the Corollary follows by Theorem 8.3. $\square$

Corollary 9.2. *Assume the situation of Corollary 9.1. Let $I' = \{1 \leq i \leq n \mid \lambda_i \neq 0\}$, and assume that the restriction of s to I' is injective. Let $V' \subseteq V$ and $W' \subseteq W$ be the Yetter–Drinfel'd submodules with bases $a_{s(i)}, i \in I'$, and $u_i, i \in I'$.*

Then

a) *$(k[\Lambda], k[\Gamma], \tau, W', V', \beta')$ is a Yetter–Drinfel'd two-cocycle twist datum. $V^\perp \subseteq W$ and $W^\perp \subseteq V$ are Yetter–Drinfel'd submodules, and the inclusion maps $W' \subseteq W$ and $V' \subseteq V$ define isomorphisms $W' \cong W/V^\perp$ and $V' \cong V/W^\perp$. The restriction*

$$\beta' : W' \otimes V' \to k$$

of β is nondegenerate.

b) *The projections $\pi_W : W \to W'$, $\pi_V : V \to V'$ define a surjective bialgebra map*

$$\left((\mathfrak{B}(W)\#k[\Lambda])\otimes(\mathfrak{B}(V)\#k[\Gamma])\right)^\sigma \xrightarrow{F} \left((\mathfrak{B}(W')\#k[\Lambda])\otimes(\mathfrak{B}(V')\#k[\Gamma])\right)^{\sigma'},$$

where

$$F = (\mathfrak{B}(\pi_W)\#\mathrm{id})\otimes(\mathfrak{B}(\pi_V)\#\mathrm{id}).$$

PROOF: By Corollary 9.1 $(k[\Gamma], k[\Lambda], \tau, W, V, \beta)$ is Yetter–Drinfel'd two-cocycle datum. Thus $(k[\Gamma], k[\Lambda], \tau, W', V', \beta')$ is Yetter–Drinfel'd two-cocycle datum as well.

To show that β' is non-singular we compute $W^\perp$ and $V^\perp$. Suppose that $a \in V$ and write $a = \sum_{j=1}^m x_j a_j$, where $x_1, \ldots, x_m \in k$. Then $a \in W^\perp$ if and only if $\beta(u_i, \sum_{j=1}^m x_j a_j) = 0$, or $\sum_{j=1}^m x_j \lambda_i \delta_{s(i),j} = 0$, for all $1 \le i \le n$. Therefore $W^\perp$ has basis $\{a_j \,|\, j \in [m]\backslash s(I')\}$. We have shown that $W^\perp \oplus V' = V$.

Let $u \in W$ and write $u = \sum_{i=1}^n y_i u_i$ where $y_1, \ldots, y_n \in k$. Then $u \in V^\perp$ if and only if $\beta(\sum_{i=1}^n y_i u_i, a_j) = 0$, or $\sum_{i=1}^n y_i \lambda_i \delta_{\sigma(i),j} = 0$, for all $1 \le j \le m$. Since s is one-one we conclude that $u \in V^\perp$ if and only if $y_i \lambda_i = 0$ for all $1 \le i \le n$. Thus $V^\perp$ has basis $\{u_i \,|\, i \in [n]\backslash I'\}$. Thus β' is non-singular, and part a) is established.

Note that the maps $\pi_W : W \longrightarrow W'$ and $\pi_V : V \longrightarrow V'$ of Yetter–Drinfel'd modules can be identified with the projections $W \longrightarrow W/V^\perp$ and $V \longrightarrow V/W^\perp$ respectively. At this point we apply Corollary 8.5 to complete the proof. $\square$

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